Tasks 05 - Single Variable Optimization

Mathematics for Master Students

These problems accompany Session 05 of the tutorial. Work through them at your own pace before the next session. We start Session 06 with your questions. On the website, every problem has a fully worked solution you can expand to check yourself. The PDF version contains only the problems, so you can print it and use it as a worksheet.

The problems are staged: the first two are warm-ups, the middle ones practice the core techniques, and the last two are at the level of the mock exam. Throughout, the recipe is always the same: differentiate, solve $f'(x) = 0$ for the candidates, then let f'' decide.

Problem 1: Find and Classify the Vertex

For each function, find the stationary point via the first-order condition $f'(x) = 0$, classify it with the second-order condition f'', and give the extreme value.

a) $f(x) = x^2 - 6x + 11$
b) $g(x) = -x^2 + 10x - 3$

Problem 2: Convex, Concave, and the Inflection Point

Consider $f(x) = x^3 - 6x^2 + 5$.

a) Find $f''(x)$.
b) On which interval is f convex, and on which is it concave?
c) Find the inflection point, where the bend switches.

Problem 3: Two Stationary Points

A cubic can have a peak *and* a valley. For $f(x) = x^3 - 6x^2 + 9x + 1$:

a) Find all stationary points via the FOC.
b) Classify each with the SOC, and give its value.
c) Is either one the *global* maximum or minimum of f on all of $\mathbb{R}$?

Problem 4: Global Optima on a Closed Interval

Find the global maximum and global minimum of $f(x) = x^3 - 12x + 5$ on the closed interval $[0, 3]$.

Problem 5: A Binding Capacity Limit

A workshop's daily profit from producing q units is

$$\pi(q) = -0.5q^2 + 60q - 400 \quad (\text{in } €) \ .$$

The plant can make at most 50 units per day, so $q \in [0, 50]$.

a) Find the unconstrained profit-maximizing quantity from the FOC.
b) Is that quantity feasible? If not, find the best feasible output and the profit there.

Problem 6: Profit Maximization from a Demand Curve

A parcel service faces the linear demand $q = 120 - 2p$, where p is the price in € and q the number of parcels. Its cost of handling q parcels is $C(q) = 10q + 300$ (€).

a) Write revenue $R(q)$ and profit $\pi(q) = R(q) - C(q)$ as functions of q.
b) Find the profit-maximizing quantity from the FOC, and confirm it with the SOC.
c) Verify that marginal revenue equals marginal cost there.
d) Find the optimal price and the maximum profit.

Problem 7: Economic Order Quantity

A distribution center faces steady annual demand of $D = 4500$ units. Each order costs a fixed $K = 60$ (€) to place regardless of size, and holding one unit in stock costs $h = 6$ (€) per year. Ordering q units at a time gives the annual total cost

$$T(q) = \frac{D}{q}K + \frac{q}{2}h.$$

a) Substitute the numbers and find the economic order quantity q^* from the FOC.
b) Confirm with the SOC that q^* minimizes cost.
c) How many orders per year is that, and what are the ordering, holding, and total costs at q^*?
d) If annual demand quadruples to $D = 18000$, what happens to q^*? Answer using the EOQ formula $q^* = \sqrt{2DK/h}$, without recomputing from scratch.