

Session 06 - Systems of Linear Equations

Mathematics for Master Students

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Recap & Your Questions

Session 05 in a Nutshell

- **First-order condition:** optima hide where $f'(x) = 0$, solve it for the candidates
- **Second-order condition:** $f''(x^*) > 0$ a minimum, $f''(x^*) < 0$ a maximum, a flat tangent alone is never enough
- On a **closed interval**, compare stationary points *and* endpoints, a boundary can win
- Profit peaks where marginal revenue = marginal cost
- The **EOQ** $q^* = \sqrt{2DK/h}$ balances ordering against holding costs

Common Issues in the Homework

- **Endpoints forgotten (P4):** on $[0, 3]$ the interior stationary point gave the *minimum* -11 ; the true maximum $f(0) = 5$ sits at the boundary, the endpoints are always candidates
- **The cap ignored (P5):** the FOC returned $q = 60$, but the plant makes at most 50. Since $\pi'(50) = 10 > 0$, profit is still rising, the best feasible output is $q = 50$, not 60
- **A sign slip in the SOC:** $f''(x^*) < 0$ is a **maximum** (concave), not a minimum, check the sign before naming the type

...

Tip

Which tasks gave you trouble? Bring them up **now**. We take the time to work through them before moving on.

Warm-Up: Recipe Check

Question

Minimize the cost $C(q) = q^2 - 8q + 20$. Where is the minimum, and how do you know it *is* a minimum?

...

- **FOC:** $C'(q) = 2q - 8 = 0 \Rightarrow q = 4$

- **SOC:** $C''(q) = 2 > 0 \rightarrow$ convex $\rightarrow$ a minimum, confirmed
- **Value:** $C(4) = 16 - 32 + 20 = 4$, lowest cost at $q = 4$

Today's Plan

- **From one equation to a system:** several conditions on several unknowns at once
- **Gaussian elimination:** a clean recipe for three unknowns
- **How many solutions?** one, none, or infinitely many
- **Linear systems in business:** equilibrium and production planning
- A look back: the toolkit you built across all six sessions

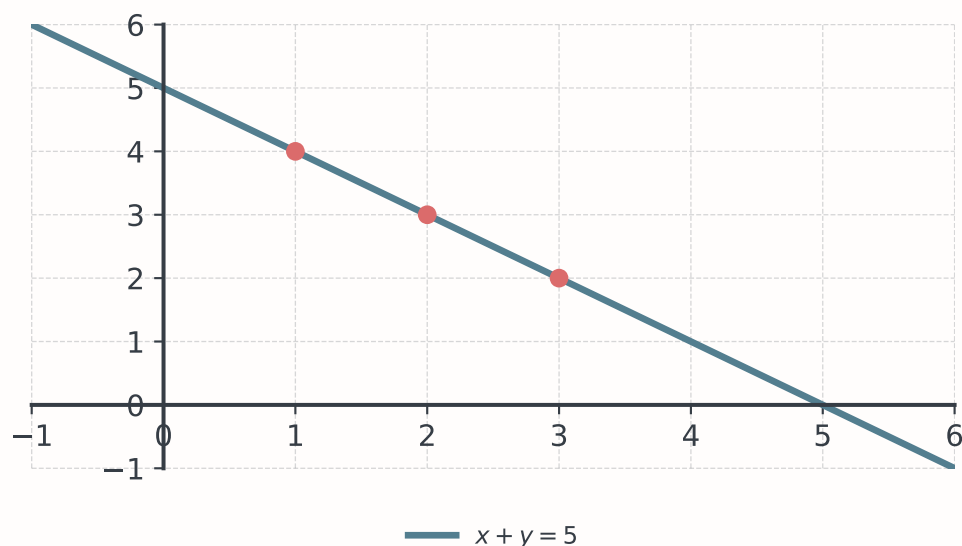
From One Equation to a System

What Makes an Equation *Linear*?

An equation is linear when every unknown appears **on its own, to the first power**, nothing more exotic.

- **Linear:** $2x + 3y = 13$, or $x_1 + 2x_2 + 2x_3 = 16$, a weighted sum of the variables
- **Not linear:** $x^2 + y = 4$ (a square), $xy = 6$ (a product), $\sqrt{x} + y = 1$ (a root)
- The rule: no powers, no products of variables, no variables under roots or in exponents
- Linear systems are the ones we can solve with pure, mechanical bookkeeping

One Equation, Two Unknowns = a Line



A single equation like $x + y = 5$ has infinitely many solutions: $(1, 4)$, $(2, 3)$, $(3, 2)$, ..., every point on one line.

A System Pins It Down

One condition leaves a whole line of choices. Add a **second** condition and the freedom shrinks.

- A system of linear equations asks several conditions to hold *at the same time*
- Business rarely has just one constraint: labour **and** material **and** budget
- Each equation is a line; a solution must sit on every line at once
- Two lines usually cross at a single point, one shared answer

A Two-Product Workshop

A furniture workshop makes **chairs** (x) and **tables** (y) and wants to use its resources fully this week:

- **Wood:** a chair takes 2 units, a table takes 3; 13 units are in stock $\Rightarrow 2x + 3y = 13$
- **Labour:** each item takes one day of work; 5 days are available $\Rightarrow x + y = 5$
- Two conditions, two unknowns, a 2×2 system:

...

$$2x + 3y = 13 \qquad x + y = 5$$

Method 1: Substitution

Solve one equation for one variable, then substitute into the other.

- From the labour equation: $x = 5 - y$
- Put that into the wood equation: $2(5 - y) + 3y = 13$
- Expand: $10 - 2y + 3y = 13 \Rightarrow 10 + y = 13$
- So $y = 3$, and back-substituting $x = 5 - 3 = 2$
- **Solution:** 3 tables and 2 chairs, i.e. $(x, y) = (2, 3)$

Method 2: Elimination

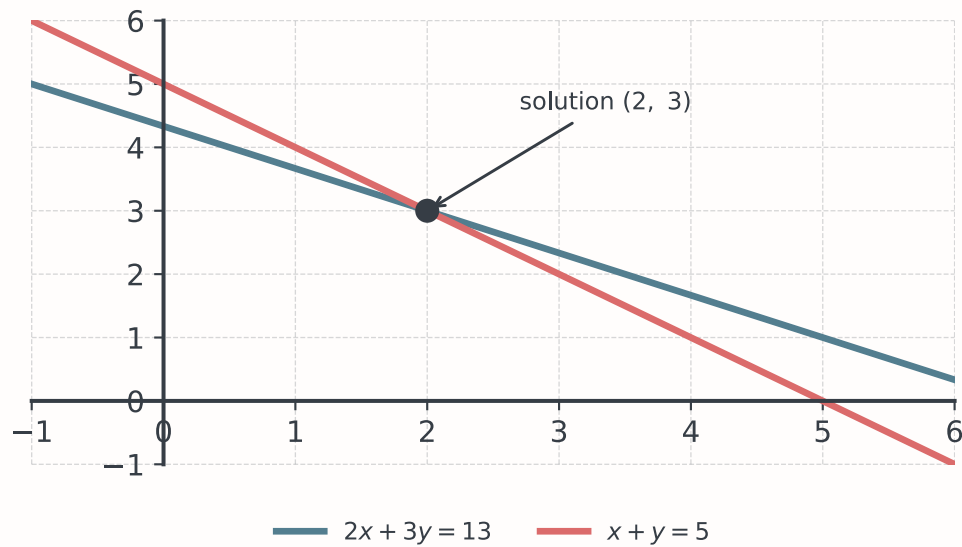
Scale the equations so one variable cancels when you subtract.

- Multiply the labour equation by 2: $2x + 2y = 10$
- Subtract it from the wood equation:

$$(2x + 3y) - (2x + 2y) = 13 - 10 \Rightarrow y = 3$$

- Back-substitute: $x + 3 = 5 \Rightarrow x = 2$
- **Same solution** $(2, 3)$, two roads, one destination

Where the Lines Meet



Always check: $2(2) + 3(3) = 4 + 9 = 13 \checkmark$ and $2 + 3 = 5 \checkmark$, the point sits on both lines.

Your Turn: Linear or Not?

i Question

Is $3xy = 12$ a linear equation in x and y ?

...

- **No**, it multiplies the two unknowns together, and xy is a product of variables
- Linear equations only allow each variable on its own, to the first power
- A quick test: could you write it as $(\text{number}) \cdot x + (\text{number})y = \text{number}$? Here you cannot

Three Unknowns: Gaussian Elimination

A Bigger System

For three unknowns, substitution gets messy. Our example for this section:

$$x + y + z = 9, \quad x + 2y + 2z = 16, \quad 2x + y + 3z = 17$$

...

Three conditions, three unknowns, and a lot of repetitive writing ahead. What we need first is a compact bookkeeping shorthand.

A Tidy Way to Write a System

The system is just its numbers, the **augmented matrix** (coefficients, then the right-hand side):

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 2 & 16 \\ 2 & 1 & 3 & 17 \end{array} \right]$$

...

Each row is an equation; the bar just separates the left from the right side. No matrix algebra, pure bookkeeping.

Three Allowed Moves

We reshape the system with three row operations, each keeps the solution unchanged:

1. **Swap** two rows (reorder the equations)
2. **Scale** a row by a non-zero number (multiply or divide an equation)
3. **Add** a multiple of one row to another (combine two equations)

...

The goal: create zeros below the diagonal, a triangular staircase we can read off from the bottom up.

Forward Elimination: Clear the First Column

Start from the augmented matrix and knock out the x -terms below the top row:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 2 & 16 \\ 2 & 1 & 3 & 17 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 1 & 7 \\ 0 & -1 & 1 & -1 \end{array} \right]$$

- $R_2 - R_1$: $(1, 2, 2 | 16) - (1, 1, 1 | 9) = (0, 1, 1 | 7)$
- $R_3 - 2R_1$: $(2, 1, 3 | 17) - 2(1, 1, 1 | 9) = (0, -1, 1 | -1)$

Forward Elimination: Clear the Second Column

Now use the new second row to clear the entry beneath it, then scale to a leading 1:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 1 & 7 \\ 0 & -1 & 1 & -1 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3/2]{R_3 \rightarrow R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

- $R_3 + R_2$: $(0, -1, 1 | -1) + (0, 1, 1 | 7) = (0, 0, 2 | 6)$
- $R_3/2$: $(0, 0, 2 | 6) \div 2 = (0, 0, 1 | 3)$

Back-Substitution

The matrix is now triangular, zeros fill the lower-left corner. Read it from the **bottom up**:

$$x + y + z = 9, \quad y + z = 7, \quad z = 3$$

- **Bottom row:** $z = 3$
- **Middle row:** $y + 3 = 7 \Rightarrow y = 4$
- **Top row:** $x + 4 + 3 = 9 \Rightarrow x = 2$
- **Solution:** $(x, y, z) = (2, 4, 3)$

Always Verify

Plug the triple $(2, 4, 3)$ back into the original three equations:

- $x + y + z = 2 + 4 + 3 = 9 \checkmark$
- $x + 2y + 2z = 2 + 8 + 6 = 16 \checkmark$
- $2x + y + 3z = 4 + 4 + 9 = 17 \checkmark$

...

Common Mistake

When you apply a row operation, act on the whole row, the right-hand side included. Subtracting $2R_1$ from R_3 but forgetting to also do $17 - 2 \cdot 9 = -1$ silently corrupts the whole solve.

Your Turn: Does Swapping Hurt?

Question

Halfway through, you **swap** two rows of the augmented matrix. Does that change the system's solution?

...

- **No**, swapping rows just reorders the equations; the set of conditions is identical
- All three row operations are *reversible* and solution-preserving
- Swapping is often useful: it can bring a convenient 1 up to the pivot spot

How Many Solutions?

One, None, or Infinitely Many

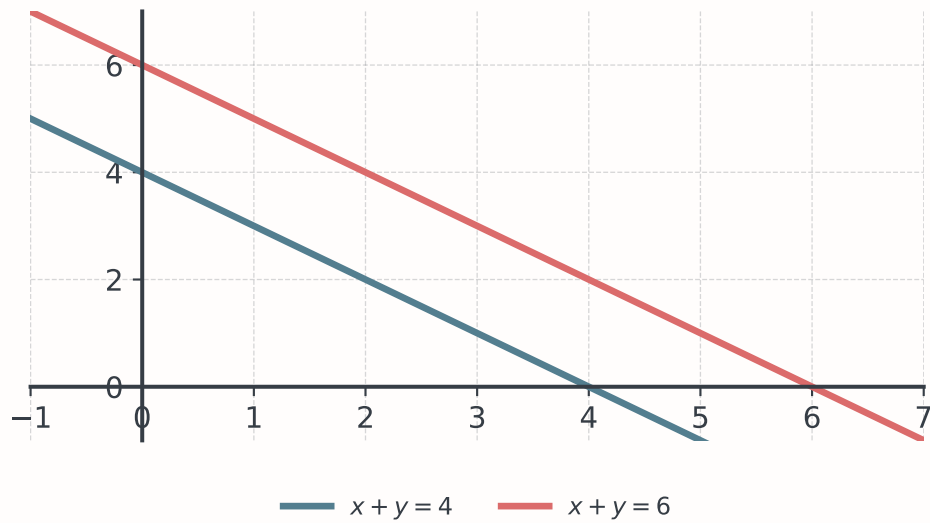
A linear system has exactly one of three fates, there is no other possibility:

- **Unique solution**, the conditions pin down one answer (our examples so far)
- **No solution**, the conditions contradict each other
- **Infinitely many solutions**, a condition is redundant, leaving freedom

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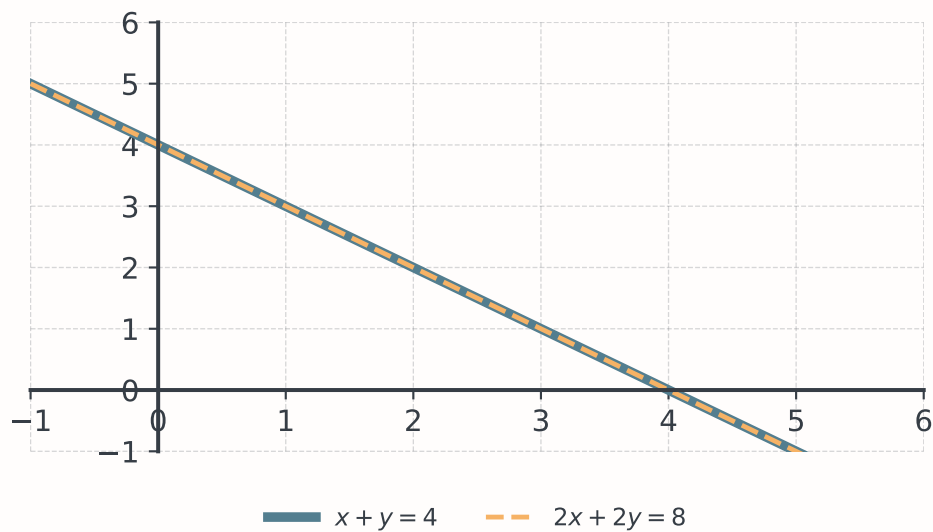
Geometry tells the story cleanly for two lines in the plane.

Two Lines: No Solution



$x + y = 4$ and $x + y = 6$ are parallel, same slope, different heights. They never meet, so **no** pair (x, y) satisfies both.

Two Lines: Infinitely Many



$2x + 2y = 8$ is just $x + y = 4$ doubled, the same line. Every point on it solves both; there are infinitely many solutions.

Parameterizing the Free Case

When one equation adds nothing new, we describe *all* solutions with a free parameter.

For $x + y = 4$ with a redundant partner:

- Let $y = t$ be free, any real number

- Then $x = 4 - t$
- **Solution set:** $(x, y) = (4 - t, t)$, one point for each t
- E.g. $t = 0 \rightarrow (4, 0)$, $t = 1 \rightarrow (3, 1)$, $t = 4 \rightarrow (0, 4)$

Three Unknowns: Planes Instead of Lines

With three unknowns, each equation is a **plane** in space, but the same three fates return:

- **Unique:** three planes meet at a single point (our Gaussian example)
- **None:** the planes have no common point, like parallel walls
- **Infinitely many:** they share a whole line (or plane) of points
- The geometry grows, but elimination handles all of it the same way

How Elimination Announces the Case

You do not need the picture, the triangular form tells you which fate you have:

- A clean staircase with a leading 1 in every column $\rightarrow$ **unique** solution
- A row like $0=5$ (all coefficients zero, non-zero right side) $\rightarrow$ **no solution**
- A row like $0=0$ (an entire row vanishes) $\rightarrow$ **infinitely many**, a free parameter

Read the Zero Row Carefully

⚠ Common Mistake

A vanished row $0 = 0$ means the equation was redundant, the system has *infinitely many* solutions, **not** “no solution”. “No solution” only appears as $0 = (\text{non-zero})$, a genuine contradiction. Read the right-hand side before you decide.

Your Turn: Reading the Outcome

i Question

During elimination a row collapses to $0 = 0$. Does the system have a unique solution, no solution, or infinitely many?

...

- $0 = 0$ is always true, that equation carried no new information
- Fewer real conditions than unknowns leaves a variable free
- So the system has infinitely many solutions, parameterize the free one

Linear Systems in Business

Market Equilibrium

A market clears where supply meets demand, quantity offered equals quantity wanted.

For a product at price p , suppose

- **Demand:** $q = 120 - 2p$, buyers want less as the price rises
- **Supply:** $q = 3p - 30$, sellers offer more as the price rises
- Equilibrium is the price and quantity solving both at once, a 2×2 system

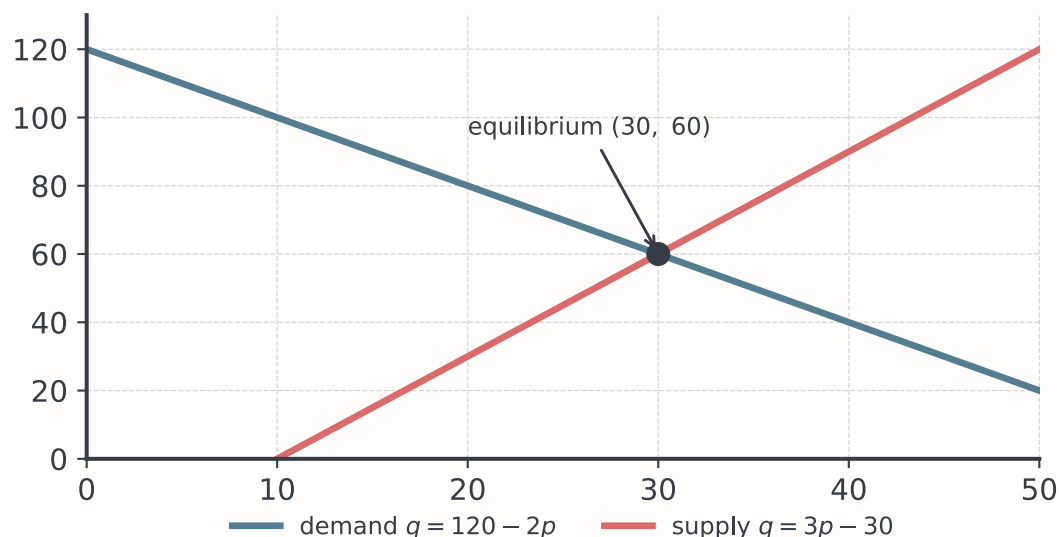
Solving for Equilibrium

Both express q , so set them equal, or eliminate q from the system

$$q + 2p = 120, \quad q - 3p = -30.$$

- Subtract the second from the first: $5p = 150 \Rightarrow p = 30$
- Back-substitute: $q = 120 - 2(30) = 60$
- **Equilibrium:** price €30, quantity 60 units
- Check the supply side: $3(30) - 30 = 60 \checkmark$

Supply Meets Demand



Below €30 buyers outnumber sellers (shortage); above it sellers outnumber buyers (surplus). The lines cross where the market balances.

Your Turn: Above the Equilibrium

i Question

The price is set at €40, above the equilibrium of €30. Is there excess supply or excess demand?

...

- Demand at €40: $120 - 80 = 40$; supply at €40: $3 \cdot 40 - 30 = 90$
- Sellers offer 90 but buyers want only 40, a surplus of 50 units
- Above equilibrium there is excess supply; price pressure pushes back down toward €30

Where the 3×3 Came From

Remember the system we solved by elimination? It was a **production plan** all along.

A workshop makes three products x, y, z and wants to use every resource fully:

- **Packaging:** one unit each, 9 available $\Rightarrow x + y + z = 9$
- **Labour:** 1, 2, 2 hours per product, 16 hours $\Rightarrow x + 2y + 2z = 16$
- **Machine:** 2, 1, 3 hours per product, 17 hours $\Rightarrow 2x + y + 3z = 17$

Reading the Production Plan

We already found the unique solution by Gaussian elimination:

$$(x, y, z) = (2, 4, 3)$$

- Make **2** of product x , **4** of product y , **3** of product z
- Every resource is used to the last unit: 9 packaging, 16 labour, 17 machine hours
- One tidy elimination turned three resource limits into a concrete production schedule
- The same recipe scales to the large planning systems behind real logistics software

Voluntary Mock Exam

To round off the tutorial, there is a voluntary mock exam covering Sessions 1–6.

- Adapted from the original KLU course's exam, real exam-style questions, no grade attached
- **Date:** to be announced, watch the tutorial page
- **How to prepare:** rework the [tasks](#), retry the self-test quizzes, and keep the cheat-sheets beside you
- Find it here: [Mock Exam](#)

Closing

Key Takeaways

- A **linear** equation is a plain weighted sum of unknowns, one such equation in two unknowns is a line
- Solve a 2×2 system by **substitution** or **elimination**, both give the same crossing point
- **Gaussian elimination** reduces a 3×3 system to triangular form, then back-substitutes
- Every system has one, none, or infinitely many solutions, the triangular form reveals which
- Equilibrium and production planning are linear systems in business dress

Skip order if running long: 1. the “Parameterizing the Free Case” slide (state the idea verbally from the coincident-lines plot), 2. “Three Unknowns: Planes Instead of Lines” (mention in one sentence), 3. the “Reading the Production Plan” detail bullets (the solution triple is the point), 4. compress the two forward-elimination slides into one if the algebra is landing.

That's it for today.

Any Questions?

Until the Mock Exam

- Work through the [Tasks](#): systems drills with worked solutions
- Check yourself with the [Self-Test](#) quiz
- Keep the [Cheatsheet](#) handy, elimination in a nutshell
- Then put it all together on the [Mock Exam](#)

The Toolkit You Built

Six sessions, one steadily growing toolkit for the math behind your M.Sc. courses:

- **Notation & logic**, then **sets**, the language to state ideas precisely
- **Functions**: how one quantity depends on another, and how to read a graph
- **Differentiation**: rates of change and the shape of a curve
- **Optimization**: turning a flat tangent into the best decision
- **Linear systems**: solving for several unknowns at once

...

You are ready for the mock exam, and for the math your degree will lean on.

Literature & Further Reading

- These sessions cover the essentials, textbooks offer more depth and practice
- **Sydsaeter & Hammond**: *Essential Mathematics for Economic Analysis*
- **Jacques**: *Mathematics for Economics and Business*
- Full recommendations on the tutorial's [literature page](#)