

Session 04 - Differentiation

Mathematics for Master Students

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Recap & Your Questions

Session 03 in a Nutshell

- A function assigns each input exactly one output
- The gallery: linear, quadratic, polynomial, rational, exponential, logarithm
- Maximal domain, red flags: division by zero, roots of negatives, logs of non-positives
- Composition chains functions (order matters); inverses undo
- Convex vs concave: where the chord lies relative to the graph
- Today: how fast does the output change, $f(a + h)$ finally pays off

Common Issues in the Homework

- **One restriction forgotten:** with $\frac{\sqrt{x+1}}{x-2}$, both the root ($x \geq -1$) and the denominator ($x \neq 2$) restrict: collect every red flag, then intersect
- **Composition order:** $(f \circ g)(x)$ means g acts **first**. Several solutions applied f first
- **The log-inequality flip:** dividing by $\ln 0.85 < 0$ flips the inequality. Dividing by a negative number always does

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Tip

Which tasks gave you trouble? Bring them up **now**. We take the time to go through them before we move on to today's topic.

Warm-Up: Compose and Restrict

Question

Let $f(x) = \sqrt{x}$ and $g(x) = 2x + 7$. Compute $(f \circ g)(1)$ and find the domain of $f \circ g$.

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- Inside out: $(f \circ g)(1) = f(g(1)) = f(9) = 3$
- The root needs $g(x) \geq 0$: $2x + 7 \geq 0$, so $x \geq -\frac{7}{2}$
- Domain of $f \circ g$: $[-\frac{7}{2}, \infty)$, the inner function must land where the outer one is defined

Today's Plan

- **Rates of change:** from secant lines to the tangent and the derivative
- **Rules of differentiation:** power, sum, product, quotient, chain, e^x and $\ln x$
- **Second derivative:** the convex/concave intuition made precise
- **Marginal analysis:** marginal cost, marginal revenue, elasticity of demand

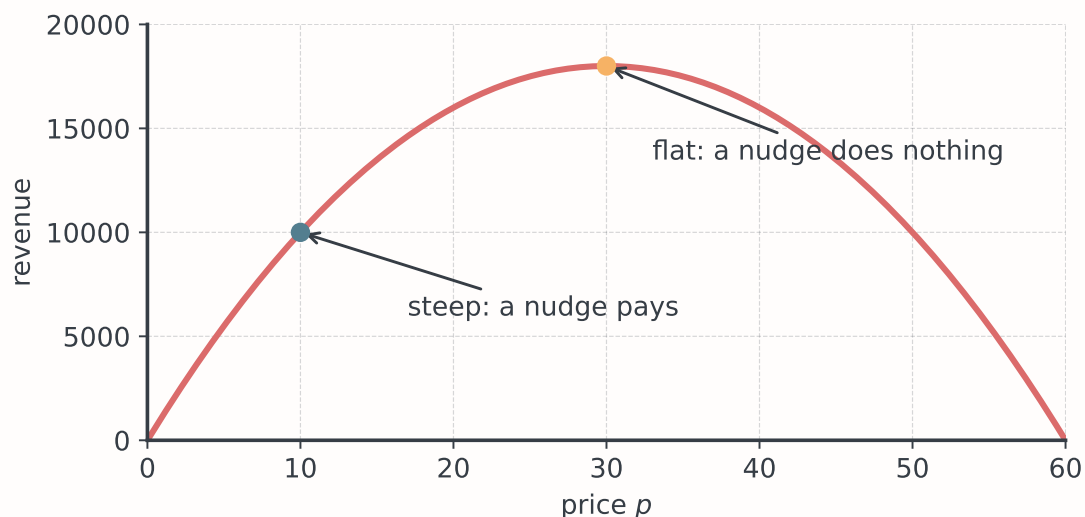
Rates of Change

How Fast Does Revenue React?

Session 3 modeled revenue as a function of price: $R(p) = 1200p - 20p^2$.

- A pricing decision is rarely “pick p from scratch”, it is nudge the current price a little
- Central question: if I move the price from p to $p + h$, how fast does revenue react?
- The answer depends on **where** we stand on the curve. Let's look

The Same Nudge, Very Different Effects



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Raising the price by €1 gains about €780 of revenue at $p = 10$, but loses about €20 at $p = 30$. One curve, very different local behavior.

The Average Rate of Change

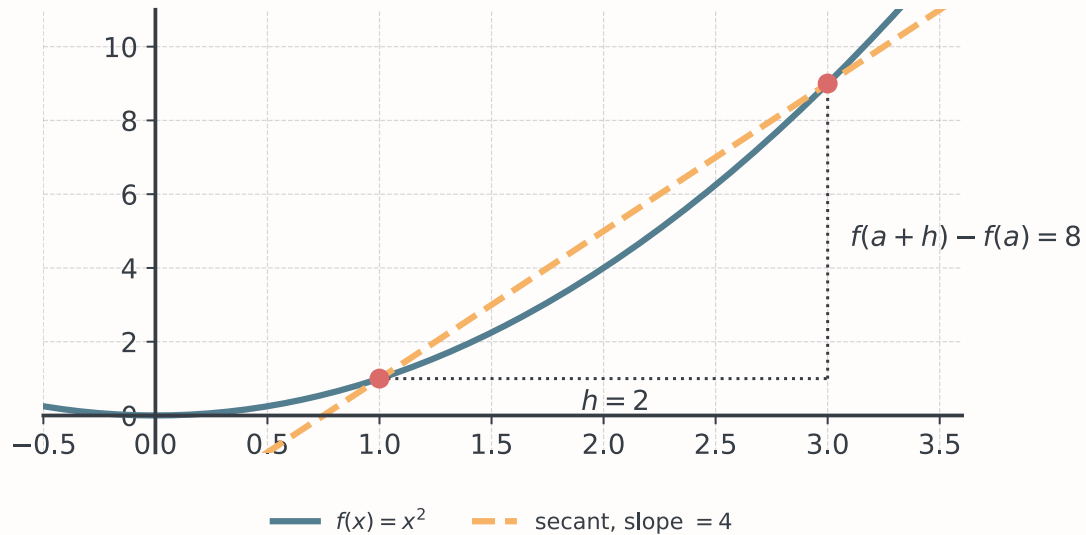
How much does f change per unit of input, between a and $a + h$?

$$\frac{f(a+h) - f(a)}{h} \quad \text{the [difference quotient] .highlight}$$

- Change in **output** divided by change in **input**, “rise over run”
- The numerator is exactly Session 3's expression $f(a+h)$ minus $f(a)$, the raw material, cashed in

- For a linear function this is always the slope m . For a curved graph it **depends on a and h**

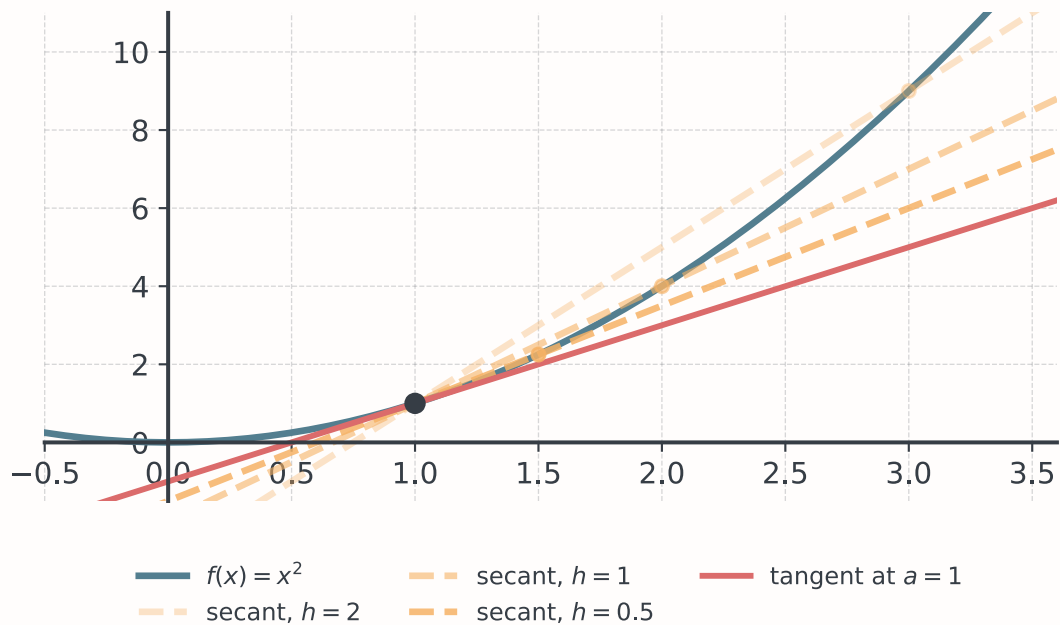
Geometry: The Secant Line



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The difference quotient is the slope of the secant line through $(a, f(a))$ and $(a + h, f(a + h))$: here $\frac{9-1}{2} = 4$ for $f(x) = x^2$ from $a = 1$ to $a + h = 3$.

From Secant to Tangent



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Keep a fixed and pull the second point closer: the secants tilt toward one limiting line, the **tangent** at a . Its slope is the rate of change *right at* a .

Letting h Shrink

For $f(x) = x^2$ at $a = 1$, the secant slope is $\frac{(1+h)^2-1}{h}$:

h	1	0.5	0.1	0.01	0.001
secant slope	3	2.5	2.1	2.01	2.001

- As h shrinks toward 0, the slopes settle on the value 2
- We write $\lim_{h \rightarrow 0}$ for “the value it settles on as h shrinks toward 0”
- That is all the limit machinery we need, no formal theory required in this tutorial

The Derivative

The derivative of f at a is the instantaneous rate of change:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

- **Geometric reading:** slope of the tangent line at $(a, f(a))$
- **Business reading:** how fast the output reacts to a tiny nudge at a
- Computed at every point, it defines a new function f' , the derivative of f
- Positive f' : increasing; negative f' : decreasing; the sign is a **trend report**

Worked Example: Differentiating $f(x) = x^2$

- **Expand** (Session 3 did this): $f(a+h) = (a+h)^2 = a^2 + 2ah + h^2$
- **Subtract:** $f(a+h) - f(a) = 2ah + h^2$
- **Divide by h :** $\frac{2ah+h^2}{h} = 2a + h$
- **Shrink h toward 0:** $f'(a) = 2a$

...

💡 Tip

Check against the table: at $a = 1$ we get $f'(1) = 2$, exactly where the secant slopes settled. And at $a = 3$: slope 6, the parabola gets steeper as we move right.

Notation for Derivatives

All of these mean the same thing:

Notation	Read as
$f'(x)$	“ f prime of x ”
$\frac{df}{dx}$	“derivative of f with respect to x ”
$\frac{d}{dx}f(x)$	“ $\frac{d}{dx}$ applied to $f(x)$ ”

...

💡 Tip

$\frac{df}{dx}$ deliberately looks like a difference quotient: it reminds you where the derivative comes from. We mostly write $f'(x)$; with several variables around (e.g. $C(q)$), the $\frac{dC}{dq}$ form says *with respect to what*.

Your Turn: Average vs Instantaneous

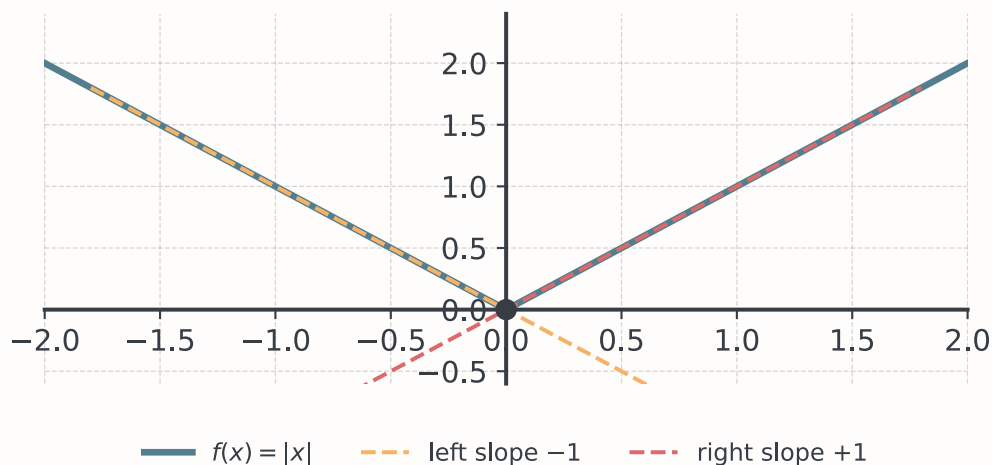
i Question

A courier van covers 150 km in 2 hours. Halfway through, a speed camera clocks it at 90 km/h. Which number is an *average* rate of change, which is a *derivative*, and can both be correct at once?

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- $\frac{150}{2} = 75$ km/h is a difference quotient, distance change over time change
- The camera reading is the instantaneous rate, the derivative of distance at that moment
- Both are correct: the average smooths out faster and slower stretches

Not Every Function Has a Derivative



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- At the kink, secants from the left settle on -1 , from the right on $+1$: no single tangent, no derivative at 0
- **Jumps** are even worse: no tangent can be drawn across a gap
- Intuition: differentiable = smooth graph: no kinks, no jumps (tiered shipping tariffs kink at the breakpoints)

Rules of Differentiation

Constants and the Power Rule

Nobody computes difference quotients all day, rules do the work:

- **Constants:** $f(x) = c$ has $f'(x) = 0$, a horizontal line has slope zero
- **Power rule:** $f(x) = x^n$ has $f'(x) = nx^{n-1}$
- Recipe: exponent down front, then reduce the exponent by one
- **Example:** $(x^5)' = 5x^4$, and our $(x^2)' = 2x$ is the case $n = 2$

The Power Rule Takes All Exponents

The rule also covers negative and fractional exponents, rewrite first:

- $\frac{1}{x} = x^{-1}$, so $(\frac{1}{x})' = -1 \cdot x^{-2} = -\frac{1}{x^2}$
- $\sqrt{x} = x^{1/2}$, so $(\sqrt{x})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

...

Tip

Session 1's exponent rules pay off: **rewrite as a power**, differentiate, translate back. Fractions and roots stop being special cases.

Sums and Constant Multiples

$$(c \cdot f)'(x) = c \cdot f'(x) \quad (f + g)'(x) = f'(x) + g'(x)$$

- Together: differentiate term by term, constants tag along
- **Example**, a cost function: $C(q) = 0.5q^2 + 10q + 200$
- $C'(q) = 0.5 \cdot 2q + 10 + 0 = q + 10$
- The fixed costs 200 vanish: they do not react to producing one unit more

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Every polynomial is now differentiable in one line. Keep $C'(q) = q + 10$ in mind, it returns later today.

The Product Rule

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

- “Derivative of the first times the second, plus the first times derivative of the second”
- **Example:** $u(x) = (x^2 + 1)(3x - 2)$
- $u'(x) = 2x \cdot (3x - 2) + (x^2 + 1) \cdot 3 = 9x^2 - 4x + 3$
- Check: expanding first gives $u(x) = 3x^3 - 2x^2 + 3x - 2$, so $u'(x) = 9x^2 - 4x + 3$ ✓

...

⚠ Common Mistake

$(f \cdot g)' \neq f' \cdot g'$, try $f(x) = g(x) = x$: the product x^2 has derivative $2x$, not 1.

The Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

- Like the product rule, but with a minus, the numerator's order matters
- **Example**, Session 3's average cost per unit: $A(q) = \frac{200+5q}{q}$
- $A'(q) = \frac{5 \cdot q - (200+5q) \cdot 1}{q^2} = -\frac{200}{q^2}$
- Always negative: average cost falls as volume grows: economies of scale, now proven

The Chain Rule

For a composition $(f \circ g)(x) = f(g(x))$, Session 3's chaining:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

- Recipe: outer derivative (inner left untouched) **times** inner derivative
- **Example**: $v(x) = (2x^2 + 1)^3$, outer $(\cdot)^3$, inner $2x^2 + 1$
- $v'(x) = 3(2x^2 + 1)^2 \cdot 4x = 12x(2x^2 + 1)^2$

...

⚠ Common Mistake

Stopping at $3(2x^2 + 1)^2$, the inner derivative $4x$ must come along. Forgetting it is the classic chain-rule slip.

Chain Rule in Business: Rates Multiply

A warehouse's stock grows over time, and storage cost depends on stock:

$$q(t) = 30 + 4t \text{ pallets after } t \text{ weeks,} \quad S(q) = 0.2q^2 \text{ cost in €}$$

...

How fast does the **cost grow over time**? Differentiate the chain $S(q(t))$:

$$\frac{dS}{dt} = S'(q) \cdot q'(t) = 0.4q \cdot 4 = 1.6(30 + 4t)$$

...

- At $t = 5$: stock $q = 50$, and cost rises at $1.6 \cdot 50 = 80$, €80 per week
- The chain rule multiplies the rates along the chain: € per pallet times pallets per week

Your Turn: Pick the Rule

i Question

Which rule differentiates each function, and what are the derivatives?

$$(a) \ (5x + 1)^4 \quad (b) \ x^3(2x - 7) \quad (c) \ \frac{x}{x^2 + 1}$$

...

- **(a) Chain rule**, outer $(\cdot)^4$, inner $5x + 1$: derivative $4(5x + 1)^3 \cdot 5 = 20(5x + 1)^3$
- **(b) Product rule**, or expand to $2x^4 - 7x^3$ and use the power rule: both give $8x^3 - 21x^2$
- **(c) Quotient rule**: $\left(\frac{x}{x^2+1}\right)' = \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

Derivatives of e^x and $\ln x$

- $(e^x)' = e^x$, the exponential is its own derivative: at every point, slope = height
- $(\ln x)' = \frac{1}{x}$, steep near 0, ever flatter afterwards: diminishing returns
- Other bases pick up a factor: $(a^x)' = a^x \ln a$
- e is exactly the base where that factor is $\ln e = 1$, the tidy one
- With the chain rule: $(e^{3x})' = e^{3x} \cdot 3 = 3e^{3x}$

The Superpower of e

Session 3's continuous growth: $K(t) = K_0 \cdot e^{rt}$. Differentiate (chain rule):

$$K'(t) = K_0 e^{rt} \cdot r = r \cdot K(t)$$

- The growth rate is proportional to the current level: at every instant, capital grows at r times itself
- That is the defining property of exponential growth: interest earns interest
- This is *why* e rules finance and growth models, the promised superpower

The Rule Table

Building blocks

$f(x)$	$f'(x)$
c	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$

Combination rules

Rule	Formula
Constant multiple	$(cf)' = cf'$
Sum	$(f + g)' = f' + g'$
Product	$(fg)' = f'g + fg'$
Quotient	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
Chain	$(f(g))' = f'(g) \cdot g'$

...

Tip

These ten lines differentiate **every function in Session 3's gallery**: practice comes with the tasks, the table lives on the cheatsheet.

Higher-Order Derivatives & Curvature

Differentiating Twice

f' is again a function, so differentiate again:

$$f''(x) = (f')'(x) \quad \text{also written} \quad \frac{d^2 f}{dx^2}$$

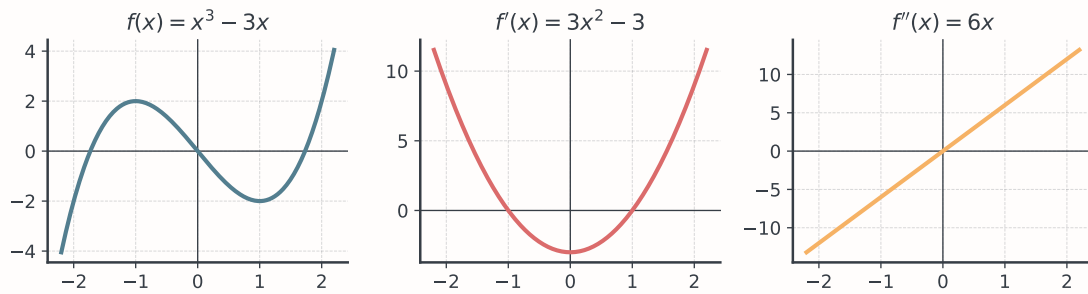
- f' reports how fast f changes; f'' reports how fast the slope changes
- **Example:** $f(x) = x^3 - 3x$, Session 3's cubic
- First: $f'(x) = 3x^2 - 3$; then: $f''(x) = 6x$
- Third, fourth, ... derivatives exist too: for us, two is all we need

Curvature: The Second-Derivative Test

Session 3's chord intuition, now with a formal test on an interval:

- $f''(x) \geq 0$ everywhere $\Rightarrow$ convex: slope increasing, graph bends **upward**, chords above
- $f''(x) \leq 0$ everywhere $\Rightarrow$ concave: slope decreasing, graph bends **downward**, chords below
- $f(x) = x^2$: $f''(x) = 2 > 0$: convex everywhere, as the chords showed
- $f(x) = \ln x$: $f''(x) = -\frac{1}{x^2} < 0$, concave everywhere: diminishing returns, certified

Reading f , f' and f'' Together



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- For $x < 0$: $f'' < 0$: slope falling, f bends downward (concave)
- For $x > 0$: $f'' > 0$: slope rising, f bends upward (convex)
- At $x = 0$ the bend switches, an inflection point; Session 5 turns these signs into an optimization tool

Your Turn: What Does $C''(q) > 0$ Mean Here?

i Question

A warehouse's handling cost $C(q)$ satisfies $C'(q) > 0$ and $C''(q) > 0$ for all realistic q . What does each sign say about the next pallet?

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- $C'(q) > 0$: handling more pallets costs more, the cost curve rises
- $C''(q) > 0$: each *additional* pallet costs more than the previous one, the rise steepens
- Business reading: the warehouse strains against capacity, the cost curve is **convex**

Marginal Analysis

Marginal Cost

Economists call $C'(q)$ the marginal cost at output level q :

- Exact meaning: the **instantaneous rate** at which cost grows at q
- Practical meaning: approximately the cost of one more unit
- Why: with $h = 1$ in the difference quotient, $C'(q) \approx \frac{C(q+1) - C(q)}{1}$
- One unit is a *small* nudge next to hundreds produced, so the tangent slope is an excellent stand-in

Worked Reading: What Does $C'(50)$ Say?

Take today's cost function $C(q) = 0.5q^2 + 10q + 200$, so $C'(q) = q + 10$.

- $C'(50) = 60$, at 50 units, cost grows at €60 per unit
- Reading: the **51st unit** costs approximately €60 to produce

- Exact check: $C(51) - C(50) = 2010.5 - 1950 = 60.5$, the approximation is off by 50 cents
- And $C''(q) = 1 > 0$: marginal cost is increasing, every further unit is dearer

...

💡 Tip

Mind the units: $C'(q)$ is **€ per unit**, not €: a rate, not a cost.

Marginal Revenue

The same idea on the revenue side, sell q units, earn $R(q)$:

- Marginal revenue $R'(q)$: approximately the revenue from **one more unit** sold
- It usually *falls* with q , selling more requires lowering the price
- Producing one more unit pays off while $R'(q) > C'(q)$, the extra revenue beats the extra cost
- Where the two rates meet lies the profit-maximal output, Session 5 makes this precise

Elasticity of Demand

Rates in **units** depend on the units, pricing needs percentages. For a demand function $D(p)$:

$$\varepsilon = \frac{p}{D(p)} \cdot D'(p)$$

- Reading: a **1% price increase** changes demand by about ε percent
- Demand falls in price, so ε is negative
- $|\varepsilon| > 1$: elastic, demand overreacts to price
- $|\varepsilon| < 1$: inelastic, demand barely reacts

Worked Example: Pricing a Delivery Service

Weekly demand for a same-day delivery service: $D(p) = 500 - 10p$, so $D'(p) = -10$.

- **At $p = 20$:** $D(20) = 300$, so $\varepsilon = \frac{20}{300} \cdot (-10) \approx -0.67$
- Inelastic: raising the price 1% loses only 0.67% of demand, revenue rises
- **At $p = 30$:** $D(30) = 200$, so $\varepsilon = -1.5$, elastic: a price increase now backfires
- The switch sits at $\varepsilon = -1$ (here $p = 25$), exactly where revenue $p \cdot D(p)$ peaks

Your Turn: Raise the Price or Not?

i Question

At the current price, a freight forwarder estimates the elasticity of demand at $\varepsilon = -0.4$. Elastic or inelastic, and what does a small price increase do to revenue?

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- $|\varepsilon| = 0.4 < 1$: inelastic, customers barely react
- A 1% price increase loses only about 0.4% of demand
- Price up, demand almost unchanged: revenue increases, typical for services customers depend on

Closing

Key Takeaways

- The derivative $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ is the instantaneous rate of change, the tangent's slope
- Kinks and jumps break differentiability, smooth graphs are safe
- Power, sum, product, quotient and chain rule handle every gallery function
- e^x is its own derivative, growth proportional to the current level
- f'' measures curvature: $f'' \geq 0$ convex, $f'' \leq 0$ concave, the chord intuition, formalized
- $C'(q) \approx$ cost of one more unit; elasticity ε compares percentage reactions

Skip order if running long: 1. The Superpower of e (fold its punchline into the $e^x/\ln x$ slide), 2. Marginal Revenue (one sentence on the worked-reading slide), 3. Not Every Function Has a Derivative (keep only the kink bullet), 4. compress The Same Nudge plot to a verbal example.

That's it for today.

Any Questions?

Until the Next Session

- Work through the [Tasks](#): the drilling happens there, with worked solutions
- Check yourself with the [Self-Test](#) quiz
- Keep the [Cheatsheet](#) next to you, the rule table lives on it
- Note down anything unclear, we start next session with **your questions**

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Tip

When differentiating, name the structure first: sum, product, quotient or chain? The right rule follows automatically.

Preview: Session 05 (Optimization)

- Central question: where is a function largest or smallest?
- The recipe: set f' to **zero** (flat tangent) and ask f'' whether it is a peak or a valley
- Best price, best order quantity, minimal cost per unit
- Today's rules become the toolkit, differentiation was the warm-up

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See you there, and bring your questions!

Literature & Further Reading

- These sessions cover the essentials, textbooks offer more depth and practice
- **Sydsaeter & Hammond:** *Essential Mathematics for Economic Analysis*
- **Jacques:** *Mathematics for Economics and Business*
- Full recommendations on the tutorial's [literature page](#)