

Session 03 - Functions

Mathematics for Master Students

Dr. Tobias Vlcek

Recap & Your Questions

Session 02 in a Nutshell

- A set is an unordered collection of distinct elements: $\in$ for elements, $\subseteq$ for sets
- $\cup, \cap, \setminus, ^c$ mirror **or, and, not**: De Morgan flips $\cup$ and $\cap$
- Counting a union: $|A \cup B| = |A| + |B| - |A \cap B|$
- $A \times B$ collects ordered pairs: $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the plane
- Constraints are sets, and they combine by **intersection**
- Today: functions map between sets, and their graphs live in $\mathbb{R}^2$

Common Issues in the Homework

- **Element vs subset**: braces build a set: $\{2\} \subseteq A$ pairs with $\subseteq$, while $2 \in A$ pairs with $\in$; mixing them was the most frequent slip
- **Complements reverse inclusion**: from $A \subseteq B$ it follows that $B^c \subseteq A^c$: writing $A^c \subseteq B^c$ is the classic inversion error
- **Mixed $\cup$ and $\cap$** : $A \cap B \cup C$ is ambiguous: grouping changes the set, so always add parentheses

...

Tip

Which tasks gave you trouble? Bring them up **now**: we take the time to go through them before we move on to today's topic.

Warm-Up: One Quick Evaluation

Question

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$. Compute $(A \cup B) \setminus (A \cap B)$.

...

- Inside out: $A \cup B = \{1, 2, 3, 4, 5\}$ and $A \cap B = \{3, 4\}$
- Removing the intersection: $(A \cup B) \setminus (A \cap B) = \{1, 2, 5\}$
- In words: everything in exactly one of the two sets: "either, but not both"

Today's Plan

- **What is a function?:** exactly one output for every input
- **Function gallery:** linear, quadratic, polynomial, rational, exponential, logarithm
- **Domain in practice:** which inputs are allowed
- **Combining functions:** composition and inverses
- **Properties:** monotone, bounded, convex vs concave

Functions as Mappings

What Is a Function?

A function $f : A \rightarrow B$ is a **rule** that assigns to each element $x \in A$ exactly one element $f(x) \in B$.

- It maps **between two sets**: the ones from last session
- Think of a *machine*: input x goes in, output $f(x)$ comes out
- “Exactly one” makes it predictable: same input, same output, every time

...

Common Mistake

“Exactly one” restricts the **outputs per input**. Two different inputs may share the same output: that is perfectly fine.

Function Notation

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto 2x + 5$$

- f is the function's **name**; x is the input, also called the argument
- $f(x)$ is the output: read “ f of x ”; here $f(x) = 2x + 5$
- $x \mapsto f(x)$ reads “ x maps to $f(x)$ ”: note the little bar on the arrow
- $f : A \rightarrow B$ names the two sets: inputs come from A , outputs land in B

...

Tip

f and $f(x)$ are different things: f is the **rule**, $f(x)$ is a **number**: the value at x .

Domain, Codomain, Range

For $f : A \rightarrow B$, three sets matter:

- The domain A : all **allowed inputs**
- The codomain B : where outputs are *declared* to land
- The range $f(A) = \{f(x) : x \in A\}$: the outputs that **actually occur**
- Always $f(A) \subseteq B$: the range is a subset of the codomain

...

Example: $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$: codomain $\mathbb{R}$, but range $[0, \infty)$: no square is negative.

Evaluating a Function

To evaluate, substitute the input everywhere the variable appears, take $f(x) = x^2 - 2x$:

- $f(3) = 3^2 - 2 \cdot 3 = 9 - 6 = 3$
- $f(0) = 0 - 0 = 0$
- $f(-1) = (-1)^2 - 2 \cdot (-1) = 1 + 2 = 3$
- Note: $f(3) = f(-1) = 3$: two inputs, same output, allowed

...

Tip

Put negative inputs in **parentheses** before squaring: $(-1)^2 = 1$, not -1 .

Plugging in Whole Expressions

The input can be an expression: substitute all of it, in parentheses. Again $f(x) = x^2 - 2x$:

$$f(a + h) = (a + h)^2 - 2(a + h) = a^2 + 2ah + h^2 - 2a - 2h$$

...

Common Mistake

$f(a + h) \neq f(a) + f(h)$, check: $f(1 + 1) = f(2) = 0$, but $f(1) + f(1) = -1 - 1 = -2$.

...

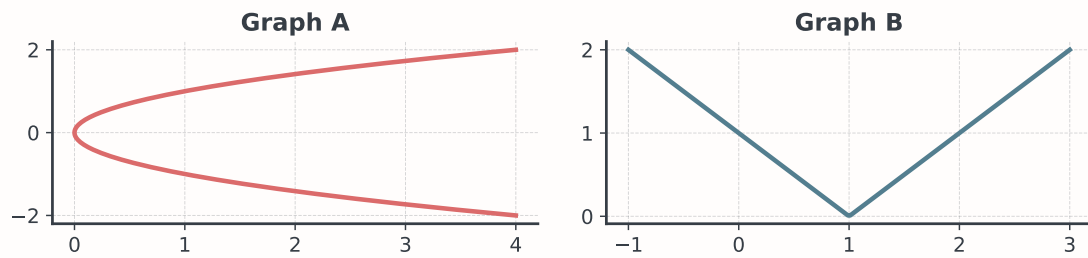
Expressions like $f(a + h)$ are the raw material of **Session 4**: “how much does f change when the input moves by h ?”

The Vertical Line Test

The graph of f is the set $\{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^2$: ordered pairs, as in Session 2.

- A curve is the graph of a function $\Leftrightarrow$ every vertical line hits it **at most once**
- Why: one input x must yield exactly one output
- The circle $x^2 + y^2 = 1$ fails: at $x = 0$ it contains *two* points, $y = 1$ and $y = -1$
- Lines, parabolas and exponential curves all pass

Your Turn: Which Graph is a Function?



i Question

Which of the two graphs shows y as a function of x ?

...

Only Graph B: A fails the vertical line test: $x = 1$ gives both $y = 1$ and $y = -1$. The kink in B is fine.

Functions in Business

Functions map decisions to outcomes, the core of every quantitative model:

- **Cost function** $C(q)$: production quantity $\rightarrow$ total cost
- **Demand function** $D(p)$: price $\rightarrow$ units sold
- **Production function**: input (labor hours) $\rightarrow$ output (units)
- One decision in, one predicted outcome out: that is why models must be functions

...

💡 Tip

Whenever a spreadsheet column is computed from another column, there is a function behind it.

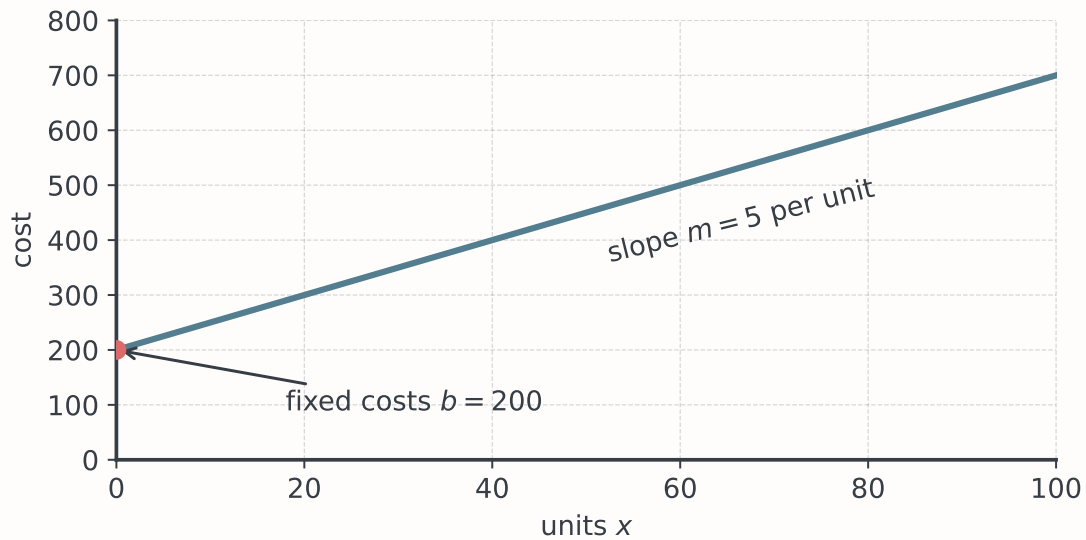
Function Gallery

Linear Functions

$$f(x) = mx + b$$

- Slope m : the output changes by m for **each unit** of input, constant everywhere
- Intercept $b = f(0)$: the starting level
- From two points on the graph: $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$
- The workhorse of business models: constant rates

A Linear Cost Function



...

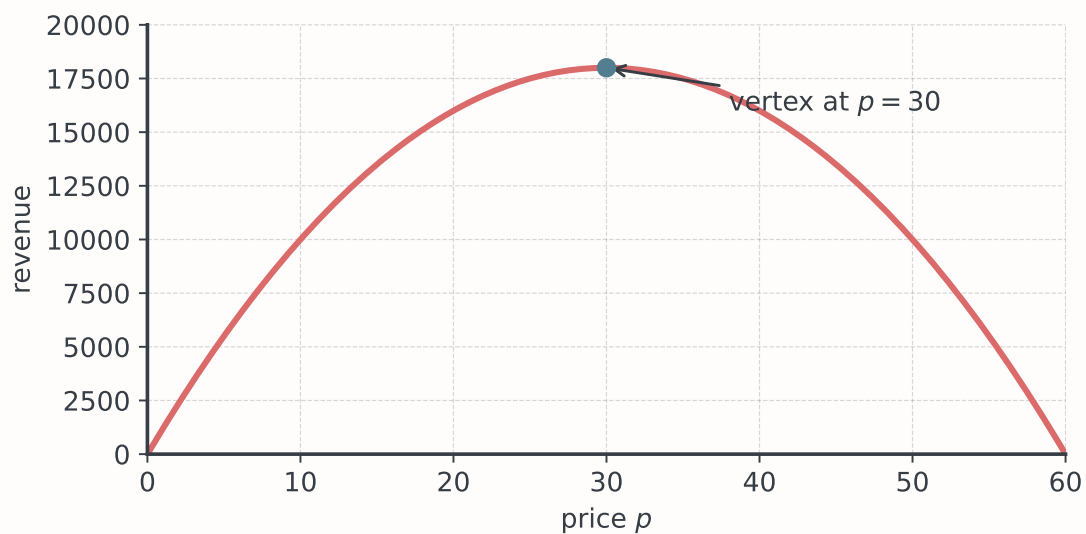
$C(x) = 200 + 5x$: the fixed costs are the intercept, the variable cost per unit is the slope.

Quadratic Functions

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

- The graph is a parabola: opens **upward** if $a > 0$, **downward** if $a < 0$
- The turning point is the vertex, located at $x = -\frac{b}{2a}$
- Downward parabolas model **trade-offs**: revenue rises with price, until demand collapses

A Quadratic Revenue Function

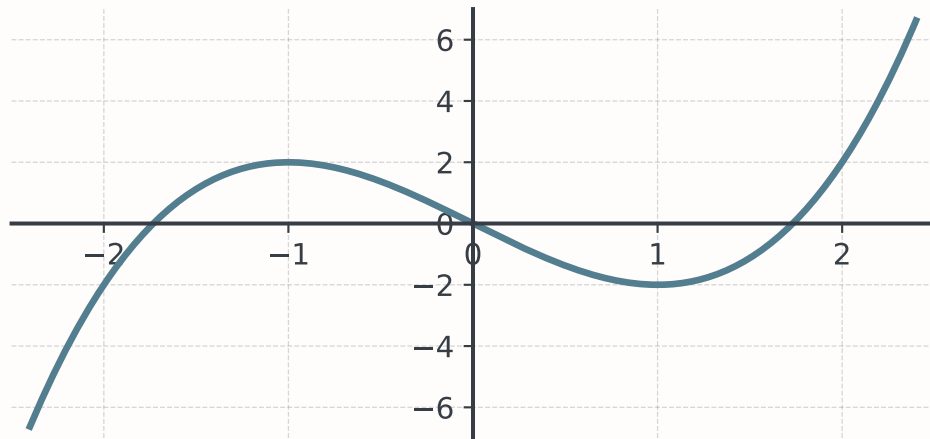


...

Demand $D(p) = 1200 - 20p$ gives revenue $R(p) = p \cdot D(p) = 1200p - 20p^2$: the vertex $p = -\frac{1200}{2 \cdot (-20)} = 30$ is the revenue-maximal price.

Polynomials

$$f(x) = a_n x^n + \dots + a_1 x + a_0 \quad \text{degree } n = \text{highest power}$$



...

Degree n allows up to n roots and up to $n - 1$ turning points: far from the origin the leading term $a_n x^n$ dominates ($f(x) = x^3 - 3x$ above behaves like x^3).

...

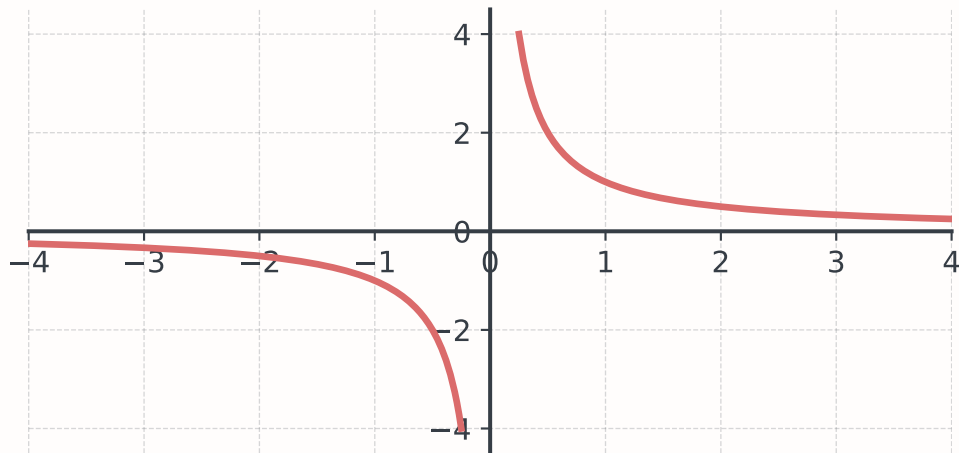
Cubic cost curves $C(x) = ax^3 + bx^2 + cx + d$ are the classic economics example: costs rise steeply, flatten, then steepen again.

Rational Functions

$$f(x) = \frac{p(x)}{q(x)} \quad \text{a quotient of two polynomials}$$

- Defined only where $q(x) \neq 0$: division by zero is the first domain trap
- Near a zero of the denominator, values explode: a **vertical asymptote**
- For large inputs, the graph can settle toward a **horizontal asymptote**

The Hyperbola $f(x) = 1/x$



...

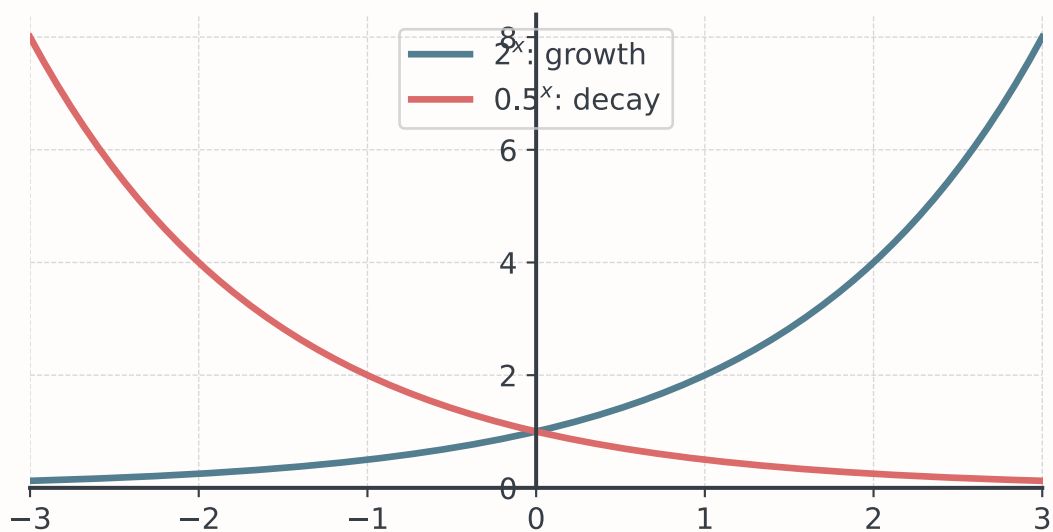
Both axes are asymptotes. **Business example:** with $C(x) = 200 + 5x$, the *cost per unit* is $\frac{C(x)}{x} = \frac{200}{x} + 5$: it falls toward 5 as volume grows: economies of scale.

Exponential Functions

$$f(x) = a \cdot b^x, \quad b > 0$$

- The **variable sits in the exponent**: each step multiplies by b
- $b > 1$: growth (interest, booming demand); $0 < b < 1$: decay (depreciation, churn)
- $a = f(0)$ is the starting value
- The standard base is Euler's number $e \approx 2.71828$: Sessions 4–5 show why it is so convenient

Growth and Decay



...

Both stay strictly positive and pass through $(0, 1)$: an exponential never reaches zero.

Compound Interest

Deposit K_0 at annual rate r , each year multiplies the balance by $(1 + r)$:

$$K_t = K_0 \cdot \underbrace{(1 + r)(1 + r) \dots (1 + r)}_{t \text{ factors}} = K_0 \cdot (1 + r)^t$$

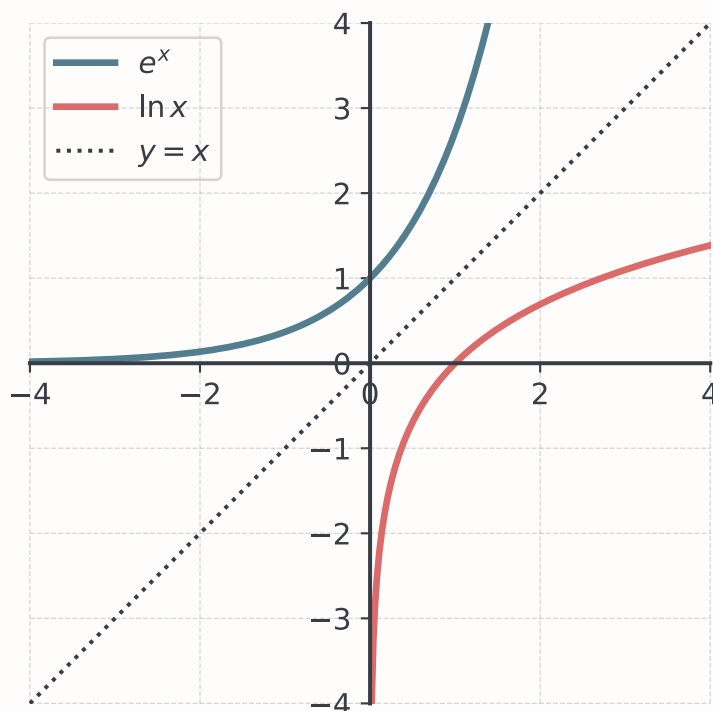
- Session 1's product idea in action: repeated multiplication becomes a power
- €1000 at $r = 0.05$: after 15 years $1000 \cdot 1.05^{15} \approx \text{€}2079$: more than **doubled**
- As a function of time t , this is exponential growth with base $b = 1.05$

Logarithms: Undoing e^x

The natural logarithm $\ln x$ answers: *which exponent produces x ?*

$$y = \ln x \Leftrightarrow x = e^y$$

- $\ln x$ is the mirror image of e^x across the diagonal
- Defined only for $x > 0$
- $\ln 1 = 0$ and $\ln e = 1$



Rules for Logarithms

$$\ln(xy) = \ln x + \ln y \quad \ln \frac{x}{y} = \ln x - \ln y \quad \ln(x^k) = k \ln x$$

- Logs turn **products into sums** and powers into multiples
- The power rule pulls the unknown **down from the exponent**: that is why logs solve growth questions
- Other bases reduce to $\ln$: $\log_b x = \frac{\ln x}{\ln b}$

...

⚠ Common Mistake

$\ln(x + y) \neq \ln x + \ln y$: the rules work for *products*, not sums.

Your Turn: When Has It Doubled?

i Question

With continuous growth at rate r , capital follows $K_t = K_0 \cdot e^{rt}$. When has it doubled?

...

$$e^{rt} = 2 \Rightarrow rt = \ln 2 \Rightarrow t = \frac{\ln 2}{r} \approx \frac{0.693}{r}$$

- At $r = 5\%$: $t \approx \frac{0.693}{0.05} \approx 13.9$ years: independent of the starting amount
- The same trick handles yearly compounding: $1.05^t = 2$ gives $t = \frac{\ln 2}{\ln 1.05} \approx 14.2$ years

Domain in Practice

The Maximal Domain

If no domain is stated, take the maximal domain: the largest set of reals on which the formula is defined.

Three red flags:

- **Division by zero**: denominators must stay $\neq 0$
- **Even roots of negatives**: under $\sqrt{}$ we need ≥ 0
- **Logs of non-positives**: inside $\ln$ we need > 0 (strictly!)

...

Everything else (polynomials, exponentials) is safe on all of $\mathbb{R}$.

Finding the Domain: Examples

Function	Restriction	Maximal domain
$\frac{1}{x-3}$	$x - 3 \neq 0$	$\mathbb{R} \setminus \{3\}$
$\sqrt{2x+6}$	$2x + 6 \geq 0$	$[-3, \infty)$

Function	Restriction	Maximal domain
$\ln(5 - x)$	$5 - x > 0$	$(-\infty, 5)$

...

💡 Tip

The results are **intervals**. Session 1's notation pays off: square bracket at -3 ($\sqrt{0}$ is fine), round bracket at 5 ($\ln 0$ is not).

Your Turn: Find the Domain

i Question

What is the maximal domain of $f(x) = \frac{\sqrt{x-2}}{x-4}$?

...

- The root needs $x - 2 \geq 0$, so $x \geq 2$
- The denominator needs $x - 4 \neq 0$, so $x \neq 4$
- **Both** must hold, intersect the conditions: $[2, 4) \cup (4, \infty)$

...

💡 Tip

Collect every restriction first, then combine: one forgotten condition is the typical exam slip.

Domains in Business

Business adds its own restrictions on top of the math:

- Quantities are **non-negative**: $q \in [0, \infty)$: no producing -50 pallets
- Capacity caps the allowed quantities: $q \in [0, 500]$
- Prices must keep demand meaningful: $D(p) = 1200 - 20p \geq 0$ forces $p \in [0, 60]$
- Model domain = mathematical domain $\cap$ business constraints: an intersection, as in Session 2

Combining Functions

Composition of Functions

Chaining two functions, apply g first, then f to the result:

$$(f \circ g)(x) = f(g(x))$$

- Read from the inside out: the inner function acts first

- **Example:** $g(x) = 2x$ and $f(x) = x + 3$: $(f \circ g)(x) = f(2x) = 2x + 3$
- But $(g \circ f)(x) = g(x + 3) = 2x + 6$: not the same
- Business reading: multi-stage processes, the output of one stage feeds the next

Order Matters: Voucher and VAT

A webshop applies a €10 voucher $v(p) = p - 10$ and 19% VAT $t(p) = 1.19p$ to a net price p :

- **Voucher first, then tax:** $(t \circ v)(p) = 1.19(p - 10) = 1.19p - 11.90$
- **Tax first, then voucher:** $(v \circ t)(p) = 1.19p - 10$
- The results differ by €1.90 on every order: for the customer, voucher first is cheaper

...

Common Mistake

Assuming $f \circ g = g \circ f$: in general, the order cannot be swapped. Always check which function acts first.

Your Turn: Compose in the Right Order

Question

Let $f(x) = x^2$ and $g(x) = x + 1$. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

...

- $(f \circ g)(2) = f(g(2)) = f(3) = 9$
- $(g \circ f)(2) = g(f(2)) = g(4) = 5$
- Inside out, always, and the order visibly changes the answer

Inverse Functions

The inverse f^{-1} **undoes** f :

$$f(a) = b \Leftrightarrow f^{-1}(b) = a$$

- It exists only if f is one-to-one: every output comes from **exactly one** input
- Graph check: every horizontal line hits the graph at most once
- $f(x) = x^2$ on $\mathbb{R}$ fails: $f(2) = f(-2) = 4$: which input produced 4?
- Restricting to $[0, \infty)$ repairs it: there $f^{-1}(x) = \sqrt{x}$

Finding an Inverse

Write $y = f(x)$, solve for x , then read the result as a function of the output.

...

Demand example: $q = D(p) = 1200 - 20p$:

$$q = 1200 - 20p \Rightarrow 20p = 1200 - q \Rightarrow p = 60 - \frac{q}{20}$$

...

- $D^{-1}(q) = 60 - \frac{q}{20}$: the price needed to sell exactly q units
- Check with a pair: $D(30) = 600$ and $D^{-1}(600) = 60 - 30 = 30$ ✓
- Domain and range swap: the domain of D^{-1} is the range of D

The Notation Trap: $f^{-1} \neq 1/f$

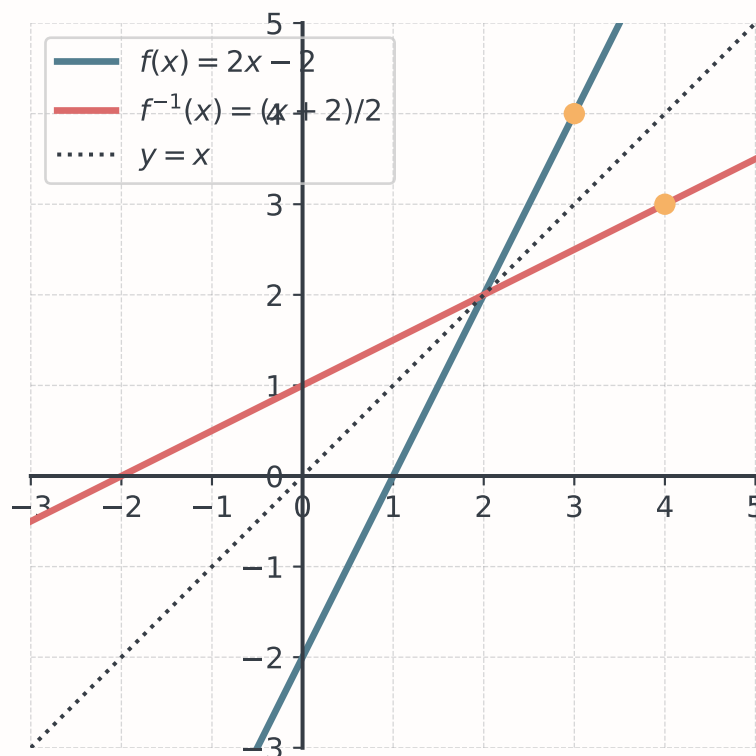
- $f^{-1}(x)$ is the **inverse function**: the -1 sits on the function *name*
- The reciprocal of the *value* is written $(f(x))^{-1} = \frac{1}{f(x)}$
- **Example** $f(x) = 2x$: $f^{-1}(x) = \frac{x}{2}$, but $\frac{1}{f(x)} = \frac{1}{2x}$: completely different

...

⚠ Common Mistake

Reading f^{-1} as “one over f ”: for functions, the exponent -1 means undo, not divide.

The Mirror Property



...

Swapping input and output reflects the graph across $y = x$: the point $(3, 4)$ on f becomes $(4, 3)$ on f^{-1} : exactly how $\ln x$ mirrored e^x .

Properties

Monotonicity

f is increasing if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$: **strictly** increasing with $<$. Decreasing works the same way, flipped.

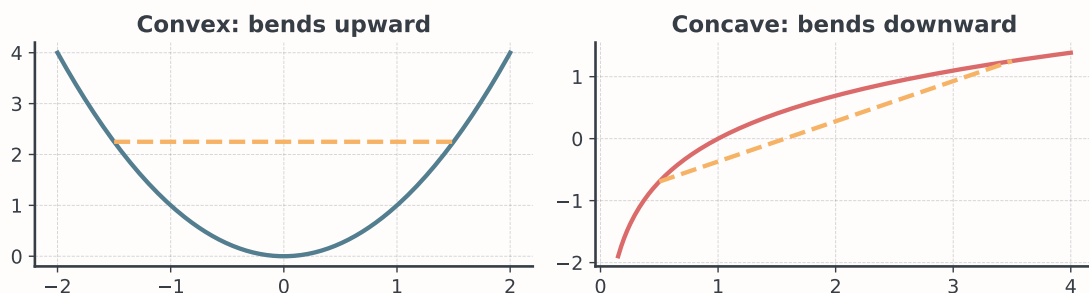
- Graph reading: strictly increasing = rises left to right, everywhere
- Costs increase in quantity; demand decreases in price
- Strictly monotone $\Rightarrow$ one-to-one $\Rightarrow$ invertible: that is why e^x has an inverse but x^2 on $\mathbb{R}$ does not

Boundedness

f is bounded above if some M satisfies $f(x) \leq M$ for **all** x : bounded below works with $\geq m$.

- $e^x > 0$: bounded **below** by 0, unbounded above
- A market share lives in $[0, 1]$: bounded on both sides
- $f(x) = x^2$: bounded below by 0, but grows without limit
- Bounds tell you what a model can, and cannot, predict

Convex or Concave?



- Convex: for two graph points, the chord lies **above the graph**: x^2, e^x
- Concave: the chord lies **below the graph**: $\ln x, \sqrt{x}$: *diminishing returns*
- Intuition only for now: Sessions 4–5 make it precise and find optima

Closing

Key Takeaways

- A function assigns each input exactly one output: domain, codomain and range are *sets*
- Know the gallery shapes: linear, quadratic, polynomial, rational, exponential, logarithm
- $\ln x$ undoes e^x : the power rule $\ln(x^k) = k \ln x$ solves for exponents
- Maximal domain: no division by zero, no even roots of negatives, no logs of non-positives

- Composition chains processes: order matters; inverses undo, but only one-to-one functions have them
- Monotone, bounded, convex/concave: reading **behavior** from a graph

Skip order if running long: 1. Mirror Property, 2. Compound Interest (keep one bullet on Growth & Decay), 3. Notation Trap (compress into Inverse Functions callout), 4. compress Functions in Business to its punchline.

That's it for today.

Any Questions?

Until the Next Session

- Work through the [Tasks](#): problems with worked solutions
- Check yourself with the [Self-Test](#) quiz
- Keep the [Cheatsheet](#) next to you while practicing
- Note down anything unclear: we start next session with **your questions**

...

Tip

When a function confuses you, **sketch it**: five plotted points reveal more than ten minutes of staring at the formula.

Preview: Session 04: Differentiation

- Central question: how fast does a function change?
- Slopes of *curved* graphs: the tangent line
- Today's $f(a + h)$ becomes the **difference quotient**
- Marginal cost and marginal revenue: the language of economic decisions
- And e^x reveals its superpower

...

See you there, and bring your questions!

Literature & Further Reading

- These sessions cover the essentials: textbooks offer more depth and practice
- **Sydsaeter & Hammond**: *Essential Mathematics for Economic Analysis*
- **Jacques**: *Mathematics for Economics and Business*
- Full recommendations on the tutorial's [literature page](#)