Session 02 - Sets

Mathematics for Master Students

Dr. Tobias Vlcek

Recap & Your Questions

Session 01 in a Nutshell

- Number sets form a hierarchy: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- Intervals like $[0, 500]$ describe connected ranges of reals
- $\sum$ and $\prod$ are compact **loops** for adding and multiplying
- Indices carry meaning: x_{ij} = flow from warehouse i to customer j
- Implication $p \Rightarrow q$ is a one-way street: only the contrapositive is equivalent
- Negation swaps quantifiers: $\neg (\forall x : P(x)) \Leftrightarrow \exists x : \neg P(x)$

Common Issues in the Homework

- **Counting terms:** $\sum_{i=3}^{10} x_i$ has $10 - 3 + 1 = 8$ terms, not 7: both endpoints count!
- **Over-strong negation:** negating "**some** warehouse operates above 90% capacity" gives "**every** warehouse runs at at most 90%", *not* "every warehouse is idle"
- **Converse error:** from "certified $\Rightarrow$ passed audit" and a supplier that passed the audit, nothing follows about its certification

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> Tip
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> Which tasks gave you trouble? Bring them up **now**: we take the time to go through them before we move on to today's topic.

Warm-Up: One More Negation

> Question
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> Negate: "**Some** delivery route is unprofitable."

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"**Every** delivery route is profitable.": negation swaps $\exists$ into $\forall$ and negates the inside.

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⚠ Common Mistake

"Some route is profitable" is **not** the negation: both statements can easily be true at the same time.

Today's Plan

- **Set basics**: elements, subsets, and cardinality
- **Set operations**: union, intersection, difference, complement
- **Products & power sets**: ordered pairs, routes, and selections
- **Sets in business**: segmentation and feasible sets

Set Basics

What Is a Set?

A set is an **unordered collection** of distinct objects, called its elements.

- Transport modes: $M = \{\text{road}, \text{rail}, \text{waterway}, \text{ocean}, \text{air}\}$
- Order does not matter: $\{1, 2, 3\} = \{3, 1, 2\}$
- No duplicates: $\{1, 2, 2, 3\} = \{1, 2, 3\}$

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Tip

Think of a set as a **bag**: what matters is only *what is inside*, not in which order it went in, and never twice.

Element Notation

Membership is the most basic set statement:

- $a \in A$: "a is an element of A"
- $a \notin A$: "a is **not** an element of A"
- $\text{rail} \in M$, but $\text{drone} \notin M$
- From Session 1: $3 \in \mathbb{N}$, while $-3 \notin \mathbb{N}$ (but $-3 \in \mathbb{Z}$)

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Every "is it in or not?" question has a clear yes/no answer: that is what makes a set well-defined.

Describing Sets

Two standard ways to write a set down:

- **Listing** the elements: $A = \{2, 4, 6, 8\}$
- **Set-builder notation**: $A = \{x \in \mathbb{N} : x \text{ is even and } x \leq 8\}$
- Read ":" as such that: some books write "|" instead

- For infinite sets, listing fails, set-builder still works: $\{x \in \mathbb{N} : x \text{ is even}\}$

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Tip

Set-builder notation is a **filter**: start from a base set, keep only the elements passing the condition, exactly what a database query does.

The Empty Set

The empty set $\emptyset = \{\}$ contains **no elements** at all.

- The set of all shipments that arrived *before* they were sent: $\emptyset$
- $\emptyset$ is a subset of **every** set: it has no element that could be missing
- An empty result is an *answer*, not an error: “no customer matches the filter”

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Common Mistake

$\emptyset \neq \{0\}$: the set *containing zero* has one element. The empty set has none.

Subsets

$A \subseteq B$: every element of A is also in B (“A is a subset of B”).

- $\{2,3\} \subseteq \{1,2,3\}$, and also $\{1,2,3\} \subseteq \{1,2,3\}$
- **Proper** subset $A \subset B$: $A \subseteq B$ *and* $A \neq B$
- Session 1's hierarchy $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ is a chain of proper subsets
- For every set: $A \subseteq A$ and $\emptyset \subseteq A$

. . .

Tip

$\subseteq$ works like $\leq$ and $\subset$ like $<$: the extra line allows equality.

Set Equality

Two sets are equal if they contain **exactly the same elements**:

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A$$

- $\{1,2,3\} = \{3,2,1\}$: same bag, different listing order
- To check equality, check **both** inclusions: a pattern you will see in many courses

Element vs Subset

The classic confusion: $\in$ relates an *element* to a set, $\subseteq$ relates *two sets*.

Let $A = \{1, 2, 3\}$:

Statement	True?	Why
$2 \in A$	✓	2 is an element of A
$\{2\} \subseteq A$	✓	every element of $\{2\}$ is in A
$\{2\} \in A$	×	the *set* $\{2\}$ is not one of the three elements

. . .

⚠ Common Mistake

Writing $2 \subseteq A$: a number is not a set, so $\subseteq$ does not apply. Ask first: element or set?

Your Turn: Element or Subset?

i Question

Let $B = \{5, 10, 15\}$. Which statements are **true**: $10 \in B, \quad \{5, 15\} \subseteq B, \quad \{10\} \in B, \quad \emptyset \subseteq B$?

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- $10 \in B$: **true**, 10 is an element
- $\{5, 15\} \subseteq B$: **true**, both elements of the left set are in B
- $\{10\} \in B$, **false**: the elements of B are numbers, not sets ($\{10\} \subseteq B$ would be true)
- $\emptyset \subseteq B$: **true**, the empty set is a subset of every set

Cardinality

The cardinality $|A|$ is the **number of elements** in A:

- $|\{\text{road}, \text{rail}, \text{waterway}, \text{ocean}, \text{air}\}| = 5$
- $|\emptyset| = 0$
- $\mathbb{N}$ has infinitely many elements: its cardinality is not a natural number

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⚠ Common Mistake

Same bars, different meaning: $|-5| = 5$ is an **absolute value** (of a number), $|\{-5\}| = 1$ is a **cardinality** (of a set). The argument decides.

Sets You Already Know

Session 1 was full of sets, we just did not call them that:

- The number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ **are** sets

- Intervals are sets in set-builder notation: $[0, 500] = \{x \in \mathbb{R} : 0 \leq x \leq 500\}$
- $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$: set-builder again
- Today we learn to combine and compare such sets

Set Operations

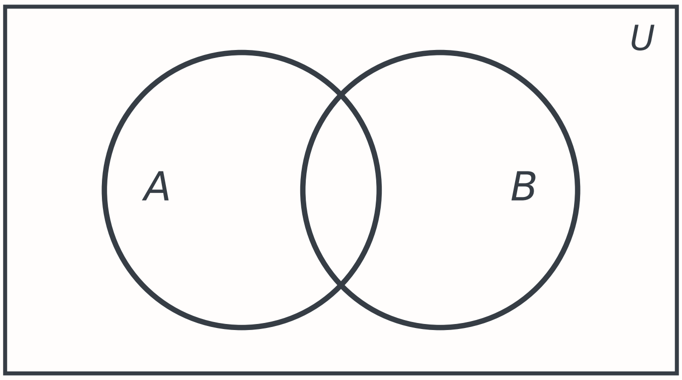
The Universal Set

Set operations play out inside a universal set U, all objects currently under discussion:

- Analyzing your customer base? U = all customers of the company
- Talking about order quantities? $U = \mathbb{R}$ or $U = \mathbb{N}_0$
- U is a **modelling choice**: state it before you start
- We need U in a moment to say what "everything *not* in A" means

Venn Diagrams

A Venn diagram shows sets as circles inside the rectangle U:



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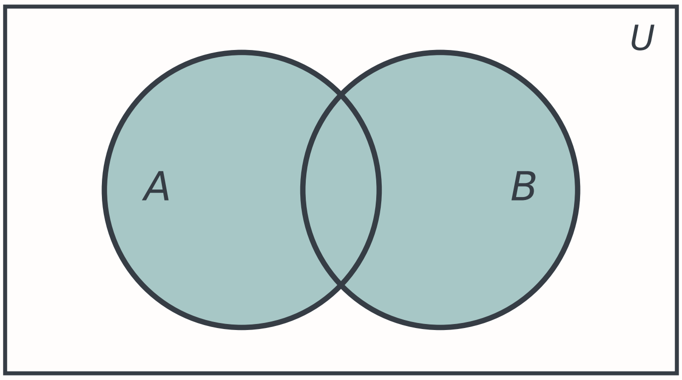
Overlap = shared elements, outside the circles = the rest of U: a thinking tool, not just a picture.

Union

$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$: everything in at least one of the sets.



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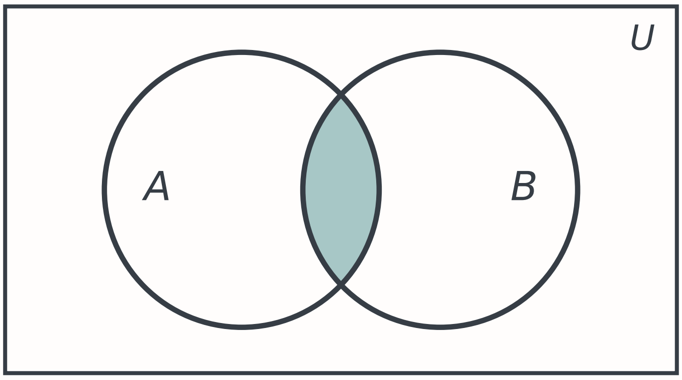
$\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$: the “or” is inclusive, just like in logic; the shared 2 appears once.

Intersection

$A \cap B = \{x : x \in A \text{ and } x \in B\}$: only the elements in both sets.



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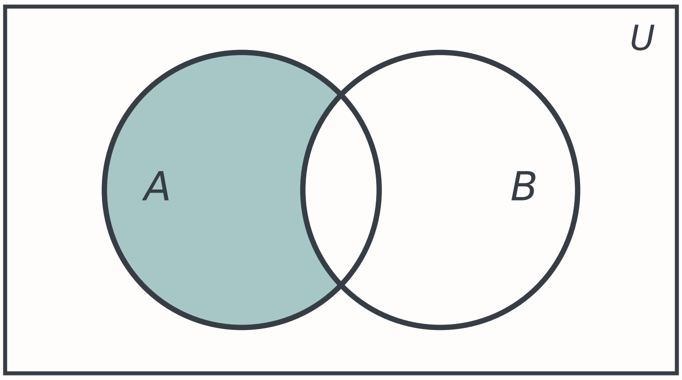
$\{1, 2\} \cap \{2, 3\} = \{2\}$: think of it as applying **two filters at once**.

Set Difference

$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$: what is left of A after removing everything in B.



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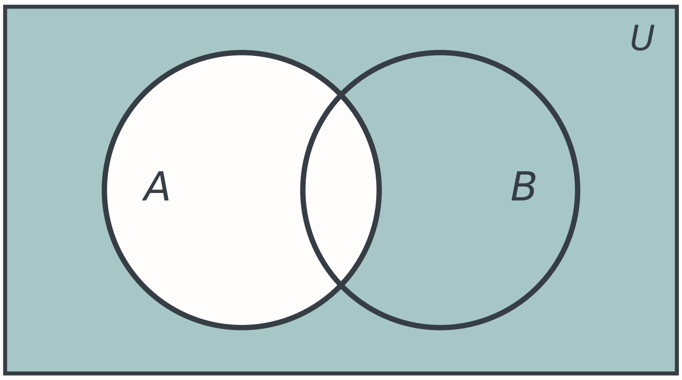
$\{1,2,3\} \setminus \{2,3\} = \{1\}$, and beware: $A \setminus B \neq B \setminus A$ in general, order matters.

Complement

$A^c = \{x \in U : x \notin A\}$: everything in U outside of A, i.e. $A^c = U \setminus A$.



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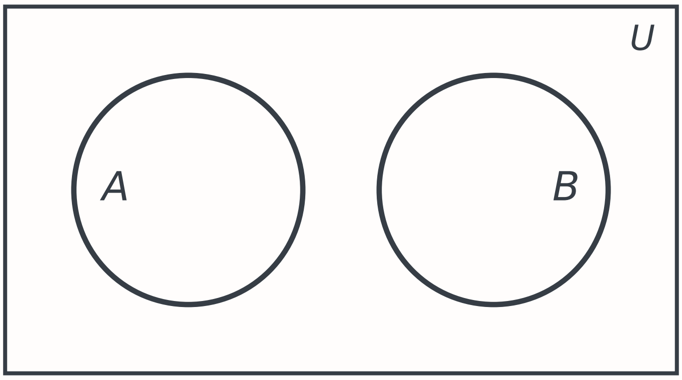
Books also write $\overline{A}$ or A': same idea. A handy identity: $A \setminus B = A \cap B^c$.

Disjoint Sets

A and B are disjoint if $A \cap B = \emptyset$, no shared elements:



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Disjoint sets are how we model **clean segmentations**: every customer is in *exactly one* region, no overlaps.

Your Turn: True or False?

ℹ Question

For any two sets A and B: is $A \cup B \subseteq A$ always true?

. . .

- **False**: the union is usually *bigger* than A
- A = premium customers, B = churned customers: $A \cup B$ contains churned non-premium customers, who are not in A
- The reverse always holds: $A \subseteq A \cup B$
- $A \cup B \subseteq A$ is only true in the special case $B \subseteq A$

How Many in the Union?

Adding $|A| + |B|$ counts the intersection twice, so subtract it once, the inclusion-exclusion rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

. . .

Survey example: of 100 coffee drinkers, 70 take sugar, 60 take cream, 50 take both:

$$|A \cup B| = 70 + 60 - 50 = 80$$

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So $100 - 80 = 20$ drink their coffee black: the complement $(A \cup B)^c$.

De Morgan's Laws for Sets

Complement flips union into intersection, and vice versa:

$$(A \cup B)^c = A^c \cap B^c \qquad (A \cap B)^c = A^c \cup B^c$$

. . .

Business example: A = delayed shipments, B = damaged shipments.

"Neither delayed nor damaged" $= (A \cup B)^c = A^c \cap B^c$ = "on time **and** intact".

. . .

> Tip
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> This is exactly Session 1's law $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$, dressed in set notation.

One Pattern, Two Languages

Set operations are logic on membership statements, $x \in A \cup B$ *means* $x \in A \vee x \in B$:

Logic (Session 1)	Sets (today)
$p \vee q$ (or)	$A \cup B$ (union)
$p \wedge q$ (and)	$A \cap B$ (intersection)
$\neg p$ (not)	A^c (complement)
$p \Rightarrow q$ (implies)	$A \subseteq B$ (subset)

. . .

> Tip
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> If you remember the logic rule, you get the set rule **for free**, and the other way around.

Products & Power Sets

Ordered Pairs

An ordered pair (a, b) is *not* a set. Here, order **does** matter:

- $(a, b) \neq (b, a)$ unless $a = b$, but $\{a, b\} = \{b, a\}$ always
- The route (Hamburg, Munich) is not the route (Munich, Hamburg)
- Coordinates in the plane are ordered pairs: $(2, 5) \neq (5, 2)$

. . .

Tip

Curly braces { } = unordered set, **round parentheses** ( ) = ordered pair. The bracket type carries the meaning.

The Cartesian Product

$A \times B = \{(a, b) : a \in A, b \in B\}$: all ordered pairs with the first entry from A, the second from B.

. . .

Example: $A = \{1, 2\}$ and $B = \{x, y\}$:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

. . .

Counting: every choice from A combines with every choice from B, so

$$|\ A \times B\ | = |\ A\ | \cdot |\ B\ |$$

Products Build Route Sets

Let $W = \{1, 2\}$ be warehouses and $C = \{1, 2, 3\}$ be customers:

- $W \times C$ = the set of all possible routes (i, j): here $2 \cdot 3 = 6$ of them
- Session 1's x_{ij} is defined **for every pair** $(i, j) \in W \times C$: the index set of the transportation problem is a Cartesian product
- $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the plane: every graph you will draw in Session 3 lives there

The Power Set

The power set $\mathcal{P}(A)$ is the set of **all subsets** of A, including $\emptyset$ and A itself.

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Example: $A = \{1, 2\}$:

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

. . .

Common Mistake

The elements of $\mathcal{P}(A)$ are themselves **sets**: $\{1\} \in \mathcal{P}(A)$, but $1 \notin \mathcal{P}(A)$.

How Many Subsets?

$$|\ \mathcal{P}(A)\ | = 2^{|\ A\ |}$$

- Why: build a subset by deciding for each element: in or out?

- Two choices per element, $|A|$ elements: $2 \cdot 2 \cdot \ldots \cdot 2 = 2^{|A|}$
- Business reading: subsets are **selections**: which suppliers to contract, which products to bundle
- That count grows *fast*: 10 candidate suppliers already give $2^{10} = 1024$ possible selections

Your Turn: Counting Options

> **i Question**
>
> A carrier may serve **any subset** of the regions $\{\text{North}, \text{East}, \text{South}\}$. How many different service portfolios are possible?

. . .

$2^3 = 8$, the elements of $\mathcal{P}(\{\text{N}, \text{E}, \text{S}\})$:

$$\emptyset, \{\text{N}\}, \{\text{E}\}, \{\text{S}\}, \{\text{N,E}\}, \{\text{N,S}\}, \{\text{E,S}\}, \{\text{N,E,S}\}$$

. . .

Note that $\emptyset$ counts: "serve no region at all" is a valid (if unambitious) portfolio.

Sets in Business

Customer Segmentation

Let P = premium customers, C = churned customers, N = newsletter subscribers:

- $P \cap C$, premium customers we **lost**: the win-back campaign list
- $C \setminus N$, churned *and* not reachable by newsletter: needs a phone call
- P^c, all non-premium customers: the upsell targets
- Every data filter you will ever build **is** a set operation: SQL's `AND`/`OR`/`NOT` are $\cap$, $\cup$, ^{c}

Reading a Set Expression

How to read $(P \cap C) \setminus N$? Work inside out:

- $P \cap C$: customers who are premium **and** churned
- … $\setminus N$: now remove everyone on the newsletter
- Result: *lost premium customers we cannot reach by newsletter*

. . .

> **⚠ Common Mistake**
>
> When $\cup$ and $\cap$ mix, grouping changes the set: $A \cup (B \cap C) \neq (A \cup B) \cap C$ in general: when in doubt, add parentheses.

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Tip

You do not need to *compute* anything to use set notation: most of its value is in making such statements **precise**.

Feasible Sets in Optimization

Every constraint defines a **set of allowed values**, and constraints combine by intersection:

- Order quantity q: capacity allows $q \leq 500$, so $q \in [0, 500]$
- The supplier requires a minimum order of 20: $q \in [20, \infty)$
- The feasible set is the intersection: $[0, 500] \cap [20, \infty) = [20, 500]$
- Optimization = picking the **best point** of the feasible set
- If the intersection is $\emptyset$, the problem is infeasible: no plan satisfies all constraints

Sets in Your M.Sc. Courses

- **Data Science:** events in probability *are* sets: $P(A \cup B)$ uses today's inclusion-exclusion rule
- **Transportation & Distribution:** decision variables are indexed by Cartesian products; feasible regions are intersections
- **Analytical Methods:** solution sets of equations and inequalities
- Whenever a slide says "for all $x \in S$", you now read it fluently

Closing

Key Takeaways

- A set is an unordered collection of distinct elements: $\in$ for elements, $\subseteq$ for sets
- $\cup$, $\cap$, ^{c} mirror **or**, **and**, **not**: De Morgan works in both worlds
- Venn diagrams turn set expressions into pictures
- $\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$: don't count the intersection twice
- $A \times B$ collects ordered pairs; $\mathcal{P}(A)$ collects subsets, $\mid \mathcal{P}(A) \mid = 2^{\mid A \mid}$
- Constraints are sets: feasibility is their intersection

That's it for today.

Any Questions?

Until the Next Session

- Work through the Tasks: problems with worked solutions
- Check yourself with the Self-Test quiz
- Keep the Cheatsheet next to you while practicing
- Note down anything unclear: we start next session with **your questions**

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Tip

Draw a Venn diagram whenever a set expression looks confusing: two circles resolve most doubts faster than staring at symbols.

Preview: Session 03: Functions

- Functions map between sets: every input from one set gets exactly one output in another
- Domain and range are, you guessed it, **sets**
- Linear, quadratic and exponential functions in business
- Graphs live in $\mathbb{R} \times \mathbb{R}$, which you met today

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See you there, and bring your questions!

Literature & Further Reading

- These sessions cover the essentials: textbooks offer more depth and practice
- **Sydsaeter & Hammond**: *Essential Mathematics for Economic Analysis*
- **Jacques**: *Mathematics for Economics and Business*
- Full recommendations on the tutorial's literature page