Cheatsheet 03 - Functions

Mathematics for Master Students

Function Basics

A **function** $f : A \to B$ assigns to each input $x \in A$ **exactly one** output $f(x) \in B$; $x \mapsto f(x)$ reads "x maps to $f(x)$". Same input, same output. Every time.

Set	Meaning	Example for $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
Domain A	All allowed inputs	$\mathbb{R}$
Codomain B	Where outputs are *declared* to land	$\mathbb{R}$
Range $f(A)$	Outputs that *actually occur*: $\{f(x) : x \in A\}$; always $f(A) \subseteq B$	$[0, \infty)$

Evaluating: substitute the input *everywhere* the variable appears, negatives and whole expressions in parentheses: $f(a + h) = (a + h)^2$, not $a^2 + h$.

Vertical line test: a curve is the graph of a function $\Leftrightarrow$ every vertical line hits it at most once.

Function Gallery

Type	Form	Shape / behavior	Business use
Linear	$mx + b$	Constant slope m; intercept $b = f(0)$	Fixed + variable cost
Quadratic	$ax^2 + bx + c$	Parabola, vertex at $x = -\frac{b}{2a}$; opens up if $a > 0$, down if $a < 0$	Revenue trade-off
Polynomial	$a_n x^n + \ldots + a_0$	Up to n roots, $n - 1$ turns; leading term dominates	Cubic cost curves
Rational	$\frac{p(x)}{q(x)}$	Undefined where $q(x) = 0$; asymptotes	Cost per unit
Exponential	$a \cdot b^x$, $b > 0$	Growth if $b > 1$, decay if $0 < b < 1$; $a = f(0)$; never reaches 0	Interest, depreciation
Logarithm	$\ln x$	Only for $x > 0$; slow growth; undoes e^x	Solving for time

Vertex form via completing the square: $ax^2 + bx + c = a(x - h)^2 + k$ with $h = -\frac{b}{2a}$, $k = c - \frac{b^2}{4a}$.

Exponent & Log Rules

Exponent rules: same base, and repeated multiplication becomes a power:

$$a^x \cdot a^y = a^{x+y} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad (a^x)^y = a^{xy}$$

Log rules: logs turn products into sums and powers into multiples:

$$\ln(xy) = \ln x + \ln y \qquad \ln\frac{x}{y} = \ln x - \ln y \qquad \ln(x^k) = k \ln x$$

$$y = \ln x \quad \Leftrightarrow \quad x = e^y \qquad \ln 1 = 0 \quad \ln e = 1 \qquad \log_b x = \frac{\ln x}{\ln b} \quad (\text{any base via } \ln)$$

Solving for the exponent: take $\ln$, pull the unknown down with the power rule:

- $1.05^t = 2 \quad \Rightarrow \quad t \ln 1.05 = \ln 2 \quad \Rightarrow \quad t = \frac{\ln 2}{\ln 1.05} \approx 14.2$
- Continuous growth $K_0 e^{rt}$ doubles at $t = \frac{\ln 2}{r} \approx \frac{0.693}{r}$ (doubling time)
- For **inequalities**: dividing by $\ln b$ with $0 < b < 1$ flips the direction, since $\ln b < 0$ (e.g. $0.85^t < 0.25 \Rightarrow t > \ln 0.25 / \ln 0.85$)

Maximal Domain: Three Red Flags

If no domain is stated, take the largest set of reals on which the formula is defined:

Red flag	Requirement	Example	Maximal domain
Division	Denominator $\neq 0$	$\frac{1}{x-3}$	$\mathbb{R} \setminus \{3\}$

Red flag	Requirement	Example	Maximal domain
Even root	Inside ≥ 0	$\sqrt{2x+6}$	$[-3, \infty)$
Logarithm	Inside > 0 (strictly!)	$\ln(5-x)$	$(-\infty, 5)$

Polynomials and exponentials are safe on all of $\mathbb{R}$. Collect **every** restriction, then intersect. Note the brackets: $\sqrt{0}$ is fine (square bracket), $\ln 0$ is not (round bracket).

Composition & Inverses

Composition: apply g first, then f; read from the inside out:

$$(f \circ g)(x) = f(g(x)) \qquad \text{order matters: } \; f \circ g \neq g \circ f \; \text{ in general}$$

Inverse f^{-1} undoes f: $f(a) = b \Leftrightarrow f^{-1}(b) = a$. It exists only if f is **one-to-one** (every horizontal line hits the graph at most once); strictly monotone $\Rightarrow$ one-to-one.

Swap-and-solve recipe: write $y = f(x)$ → solve for x → read the result as a function of the output.

- Example: $y = 2x - 2 \quad \Rightarrow \quad x = \frac{y+2}{2}$, so $f^{-1}(x) = \frac{x+2}{2}$
- Domain and range **swap**; the graph of f^{-1} is the mirror image of f across $y = x$

> ⚠ Notation Trap
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> $f^{-1}(x)$ is the **inverse function**, not $\frac{1}{f(x)}$. For functions, the exponent -1 means *undo*, not *divide*.

Properties

Property	Definition	Examples
Increasing	$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ (strictly: $<$)	e^x, costs in quantity
Decreasing	$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ (strictly: $>$)	0.85^x, demand in price
Bounded	Above: $f(x) \leq M$ for all x; below: $f(x) \geq m$	$e^x > 0$: bounded below
Convex	Chord between two graph points lies **above** the graph	x^2, e^x
Concave	Chord lies **below** the graph: diminishing returns	$\ln x$, $\sqrt{x}$

A straight line is both convex and concave (chords lie *on* the graph). Strictly monotone $\Rightarrow$ invertible: that is why e^x has an inverse but x^2 on $\mathbb{R}$ does not.

Common Pitfalls

> ⚠ Watch Out
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> - $f(a+h) \neq f(a) + f(h)$: substitute the **whole** expression, in parentheses.
> - $\ln(x+y) \neq \ln x + \ln y$: the log rules work for *products*, not sums.
> - $f \circ g \neq g \circ f$ in general: always check which function acts **first**.
> - $\ln$ needs > 0 **strictly**; an even root is happy with ≥ 0 (round vs square bracket).
> - Two inputs may share one output, but one input can **never** have two outputs.