Cheatsheet 02 - Sets

Mathematics for Master Students

Set Notation

A **set** is an unordered collection of distinct elements: $\{1, 2, 3\} = \{3, 1, 2\}$ and $\{1, 2, 2\} = \{1, 2\}$.

Symbol	Meaning	Example
$a \in A$	a is an element of A	$5 \in \{1, 5, 9\}$
$a \notin A$	a is not an element of A	$7 \notin \{1, 5, 9\}$
$\emptyset$	Empty set: no elements	$\emptyset = \{\}, \mid \emptyset \mid = 0$
$\mid A \mid$	Cardinality: number of elements	$\mid \{1, 5, 9\} \mid = 3$

Two ways to describe a set: listing $A = \{2, 4, 6\}$, or **set-builder** (a filter: base set + condition, ":" reads *such that*):

$$\{x \in \mathbb{N} : x \leq 4\} = \{1, 2, 3, 4\} \qquad [0, 500] = \{x \in \mathbb{R} : 0 \leq x \leq 500\}$$

Element vs subset: $\in$ relates an *element* to a set, $\subseteq$ relates *two sets*. For $A = \{1, 2, 3\}$:

Symbol	Meaning	Example (true)
$\in$	Element of	$2 \in A$
$\subseteq$	Subset (equality allowed, like $\leq$)	$\{1, 3\} \subseteq A$, $A \subseteq A$, $\emptyset \subseteq A$
$\subset$	Proper subset ($\subseteq$ and $\neq$, like $<$)	$\{1, 3\} \subset A$, but **not** $A \subset A$

Set equality: $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. Check **both** inclusions.

Set Operations

All inside a **universal set** U (state it first!). Tiny examples with $A = \{1, 4\}$, $B = \{4, 7\}$, $U = \{1, 4, 7, 9\}$:

Symbol	Name	Definition	Example
$A \cup B$	Union ("or")	$\{x : x \in A \text{ or } x \in B\}$	$\{1, 4, 7\}$
$A \cap B$	Intersection ("and")	$\{x : x \in A \text{ and } x \in B\}$	$\{4\}$
$A \setminus B$	Difference	$\{x : x \in A \text{ and } x \notin B\}$	$\{1\}$
A^c	Complement	$\{x \in U : x \notin A\} = U \setminus A$	$\{7, 9\}$

A and B are **disjoint** if $A \cap B = \emptyset$. Order matters in differences: $A \setminus B \neq B \setminus A$ in general.
De Morgan's laws: the complement flips $\cup$ into $\cap$ and vice versa:

$$(A \cup B)^c = A^c \cap B^c \qquad (A \cap B)^c = A^c \cup B^c$$

Distributive law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Inclusion-exclusion: don't count the intersection twice:

$$\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$$

Complement & Inclusion

Complement **reverses** inclusion (the set version of the contrapositive):

$$A \subseteq B \quad \Rightarrow \quad B^c \subseteq A^c$$

> ⚠ Wrong Direction
>
> $A \subseteq B \Rightarrow A^c \subseteq B^c$ is **false** in general. The *smaller* set has the *bigger* complement.

Products & Power Sets

Ordered pair (a, b): order matters. $(a, b) \neq (b, a)$ unless $a = b$; but $\{a, b\} = \{b, a\}$ always.

$$A \times B = \{(a,b) : a \in A, b \in B\} \qquad |A \times B| = |A| \cdot |B|$$

Power set $\mathcal{P}(A)$ = the set of **all subsets** of A, including $\emptyset$ and A itself:

$$\mathcal{P}(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\} \qquad |\mathcal{P}(A)| = 2^{|A|}$$

Its elements are themselves sets: $\{a\} \in \mathcal{P}(A)$, but $a \notin \mathcal{P}(A)$.

Logic ↔ Sets Dictionary

Set operations are logic on membership statements, one pattern, two languages:

Logic (Session 1)	Sets (Session 2)
$p \vee q$ (or)	$A \cup B$ (union)
$p \wedge q$ (and)	$A \cap B$ (intersection)
$\neg p$ (not)	A^c (complement)
$p \Rightarrow q$ (implies)	$A \subseteq B$ (subset)
$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	$(A \cup B)^c = A^c \cap B^c$

Common Pitfalls

⚠ Watch Out

- $\emptyset \neq \{0\}$: the empty set has **no** elements; $\{0\}$ has one.
- $\{2\} \in \{1,2,3\}$ is **false**, $\{2\} \subseteq \{1,2,3\}$ is true. Element or set, decide first.
- Adding $|A| + |B|$ double-counts $A \cap B$. Always subtract it once.
- A complement A^c is meaningless until the **universal set** U is stated.
- Mixed $\cup$ and $\cap$ need **parentheses**: $A \cap (B \cup C) \neq (A \cap B) \cup C$ in general.